Published for SISSA by Springer

Received: *August 19, 2025*
Accepted: *October 5, 2025*
Published: *November 6, 2025*

# Planons and their Carroll-Galilei symmetries

**José Figueroa-O'Farrill**,[a] **Simon Pekar**,[b,c] **Alfredo Pérez**[d,e,g] **and Stefan Prohazka**[f,g]

[a] *Maxwell Institute and School of Mathematics, The University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh EH9 3FD, Scotland, United Kingdom*
[b] *International School for Advanced Studies (SISSA), Via Bonomea 265, 34136 Trieste, Italy*
[c] *Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Via Valerio 2, 34127 Trieste, Italy*
[d] *Centro de Estudios Científicos (CECs), Avenida Arturo Prat 514, Valdivia, Chile*
[e] *Facultad de Ingeniería, Universidad San Sebastián, sede Valdivia, General Lagos 1163, Valdivia 5110693, Chile*
[f] *Faculty of Physics, Mathematical Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria*
[g] *Erwin Schrödinger Int. Institute for Mathematics and Physics, University of Vienna, Boltzmanngasse 9, 1090 Vienna, Austria*

*E-mail:* j.m.figueroa@ed.ac.uk, spekar@sissa.it, alfredo.perez@uss.cl, stefan.prohazka@univie.ac.at

Abstract: We study the dynamics of planons, particles whose mobility is restricted to a plane, through the classification of coadjoint orbits and unitary irreducible representations of the centrally extended planon group. Planons are closely related to Galilei/Bargmann symmetries and, remarkably, the often-ignored massless coadjoint orbits of the Galilei group play a central rôle in their description. We thereby provide a nontrivial physical interpretation of these orbits. We further construct classical and quantum dipoles as bound states of monopoles, where the restricted planar motion arises from a novel mixed Carroll-Galilei symmetry. We also argue that the simplest and already experimentally realised systems with Carroll symmetry are in crystals.

Keywords: Global Symmetries, Space-Time Symmetries

ArXiv ePrint: 2508.08390



https://doi.org/10.1007/JHEP11(2025)030

JHEP11(2025)030

## Contents


**1 Introduction** **1**

**2 Classical planon particles** **5**
2.1 The central extension of the planon group 6
2.2 Coadjoint orbits 7
2.3 Coadjoint orbits for $n = 3$ 9

**3 Elementary particles and mobility** **12**
3.1 Maurer-Cartan one-form 13
3.2 Geometric particle dynamics 14
3.3 Particle dynamics 15

**4 Composite dipoles** **22**
4.1 Dynamics of non-elementary dipoles from symmetries 22
4.2 Composite dipole from the interaction of two elementary monopoles 24
4.3 Symmetries of the composite dipole 25
4.4 Coupling of the dipole to a traceless fracton gauge field 27
4.5 First quantisation of the composite dipole with fracton gauge field 28
4.6 Symmetries of the free planon field theory 30

**5 Quantum planon particles** **33**
5.1 UIRs for characters in point-like orbits 33
5.2 UIRs for characters in one-dimensional orbits 35
5.3 Summary 38

**6 Discussion** **39**

**A Summary of [1] in planon language** **41**

**B Central extensions** **43**

**C The extended planon group is a regular semidirect product** **44**


## 1 Introduction

The Galilei algebra and its centrally extended version, the Bargmann algebra, play a central role in physics. They describe the kinematics of particles moving at velocities much smaller than the speed of light. From the perspective of group theory, such non-relativistic particles are described by coadjoint orbits or, in the quantum case, by unitary irreducible representations of the Bargmann group, where the mass is non-vanishing [2]. Surprisingly, the Galilei/Bargmann group admits various other coadjoint orbits and unitary irreducible representations (see, e.g., [1] for a complete classification), most of which lack a clear physical interpretation,



JHEP11(2025)030

even though they are mathematically well-defined (see, e.g., [3] for a particular application to geometrical optics).

In this work, we show that the coadjoint orbits and unitary irreducible representations of the Bargmann group, which are often ignored in applications in nonrelativistic physics, play a key role in the description of certain systems with restricted mobility. These so-called planons belong to the class of fracton models [4–7]. Furthermore, systems with these symmetries are, via duality to elasticity in $2+1$ dimensions, experimentally realised [8].[1]

Systems with immobility or restricted mobility have shown up at various corners of physics with possible applications ranging from quantum memory [4–7], to (conformal) carrollian symmetries and their relation to asymptotically flat spacetimes [9–13]. Due to the restricted mobility, standard quantum field theories results often do not apply. This therefore presents novel challenges and consequently new opportunities to sharpen our understanding of quantum field theories [14, 15]. One intriguing question arises: *What defines these novel particles with restricted mobility, and how can they be coupled to other systems?*

In this work we answer this question for particles with conserved electric charge, but what might be less familiar, also conserved dipole moment and trace (of quadrupole) charge

$$Q = \int d^3x\rho \qquad \boldsymbol{D} = \int d^3x\rho\boldsymbol{x} \qquad Z = \frac{1}{2}\int d^3x\,\rho\,\|\boldsymbol{x}\|^2 . \tag{1.1}$$

These charges can be derived as a consequence of the following current conservation and suitable fall-offs for the fields towards spatial infinity

$$\dot{\rho} + \partial_i\partial_j J^{ij} = 0 \qquad \delta_{ij}J^{ij} = 0 . \tag{1.2}$$

To build some intuition let us first consider a charged isolated monopole with trajectory $\boldsymbol{z}(t)$ and charge density $\rho = q\delta(\boldsymbol{x} - \boldsymbol{z}(t))$. Since $q$ is constant, the charge $Q = q$ is trivially conserved, however the conservation of the dipole moment, $\dot{\boldsymbol{D}} = q\dot{\boldsymbol{z}}(t) = \boldsymbol{0}$, puts restrictions on the mobility. Hence, isolated monopoles are immobile which is the characteristic feature of fractons [4–6].

What about dipoles? The charge density of a dipole with a constant dipole moment $\boldsymbol{d}$ is given by $\rho = -d^i\partial_i\delta(\boldsymbol{x} - \boldsymbol{z}(t))$ and leads to the following charges

$$Q = 0 \qquad \boldsymbol{D} = \boldsymbol{d} \qquad Z = \boldsymbol{d}\cdot\boldsymbol{z}(t) . \tag{1.3}$$

As expected, the charge vanishes and the dipole moment is conserved, but the trace charge puts restrictions on the mobility

$$\dot{Z} = \boldsymbol{d}\cdot\dot{\boldsymbol{z}}(t) = 0 . \tag{1.4}$$

From this condition, it is clear that the conservation of the trace charge implies that the velocity along the direction of the dipole moment must vanish. Thus the dipole can move only in the plane orthogonal to its dipole moment, maintaining its orientation, see figure 1.

[1]The relation is between fractons and crystalline defects. The immovable fractons (or monopoles) correspond to disclinations, whereas the partially movable dipoles correspond to dislocations. The allowed motion which is restricted to be orthogonal to the dipole moment is then related to the glide constraint in crystals which allows dislocations to move only in the direction of the Burgers vector.



JHEP11(2025)030

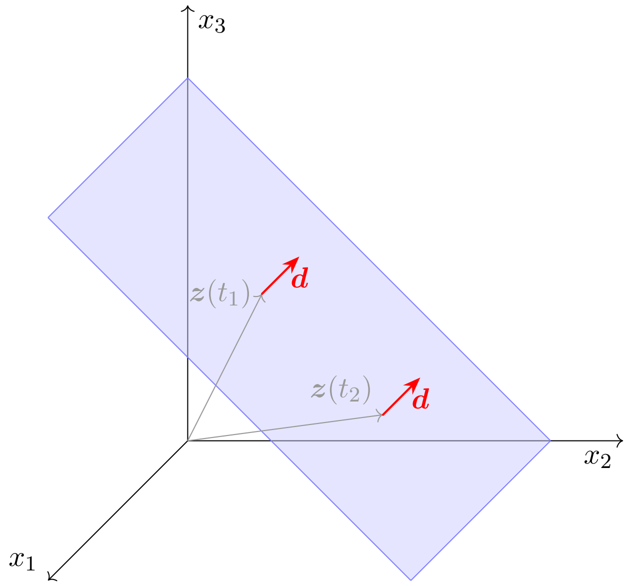


**Figure 1.** Due to the conservation of the trace charge $\dot{Z} = \boldsymbol{d} \cdot \dot{\boldsymbol{z}}\,(t) = 0$ planons with dipole moment $\boldsymbol{d}$ are allowed to move but are restricted to the plane transversal to the dipole moment.

This distinctive characteristic feature justifies why they are referred to as "planons." As we will explain, planons have an interesting mixed Carroll-Galilei symmetry. Intuitively the particle Lagrangian is invariant under galilean boosts in the transverse direction, where it can move freely, and under carrollian boosts in the direction in which its motion is restricted.

The whole system also possesses rotational, time and spatial translational symmetries, whose generators we denote by $L_{ab}$, $H$, $P_a$. The planon algebra, which is an instance of the multipole algebra [16], is then spanned by $\{L_{ab}, P_a, Q, D_a, Z := \delta^{ab} Q_{ab}, H\}$. There are no boost generators, since like for fractons, the geometry is aristotelian [17, 18]. The non-vanishing brackets are

$$[L_{ab}, L_{cd}] = \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc} \tag{1.5a}$$
$$[L_{ab}, P_c] = \delta_{bc} P_a - \delta_{ac} P_b \tag{1.5b}$$
$$[L_{ab}, D_c] = \delta_{bc} D_a - \delta_{ac} D_b \tag{1.5c}$$
$$[P_a, Z] = D_a \tag{1.5d}$$
$$[P_a, D_b] = \delta_{ab} Q\,, \tag{1.5e}$$

where $H$ is central. As already emphasised by Gromov [16], this algebra is closely related to the centrally extended Galilei algebra, i.e., the Bargmann algebra. We have summarised their correspondence in appendix A and table 4, and at various instances we will use or contrast our results to this case [1].

The classical elementary particles of a Lie group $G$ are its homogeneous symplectic manifolds. Roughly speaking, these are the coadjoint orbits of one-dimensional central extensions of $G$. In section 2.1 we exhibit the unique (nontrivial and up to equivalence) one-dimensional central extension of the planon group. At the Lie algebra level, this consists in adding a new generator $W$ to the above basis for the planon algebra and modifying the

Lie brackets in equation (1.5) by the addition of

$$[H, Z] = W\,. \tag{1.6}$$

In section 2.2 we study the coadjoint representation of the centrally extended planon group or, equivalently, the action of the centrally extended planon group on the conserved charges. We then restrict our attention to 3 spatial dimensions and classify the coadjoint orbits in section 2.3. These are summarised in table 1.

In section 3 we discuss the mobility of the classical particles. We start in section 3.1 with a review of how to associate a particle trajectory with a coadjoint orbit. We use the data defining the coadjoint orbit to define a variational problem for curves in the Lie group: curves which we can then map to particle trajectories on a chosen homogeneous spacetime of the Lie group. This is done for each of the coadjoint orbits in section 3.2 and the results are summarised in table 2. We then concentrate on those coadjoint orbits of the unextended planon group (the other orbits lead to trajectories where energy is unbounded from below) and study their dynamical systems in more detail in section 3.3. We indeed find a monopole (and its spinning counterpart) from first principles

$$S_{\text{mono}}\,[\boldsymbol{x}, \boldsymbol{\pi}] = \int dt[\boldsymbol{\pi} \cdot \dot{\boldsymbol{x}}]\,. \tag{1.7}$$

As expected, the equation of motion restricts the mobility to $\dot{\boldsymbol{x}} = \mathbf{0}$. It is interesting to note that this particle is related to the well-known massive Galilei particle, where the immobility translates into the constancy of velocity (see section 3.3.4 for details). We also find elementary dipoles with movement restricted to a plane. They are related to the often ignored massless Galilei orbits.

In section 4 we construct composite dipoles from the bound state of two monopoles, leading to the action

$$S_{\text{dip}}[\boldsymbol{z}, \boldsymbol{\pi}, \boldsymbol{\sigma}, \boldsymbol{d}] = \int dt \left[\pi_i \dot{z}^i + \sigma_i \dot{d}^i - H\right] \qquad\qquad H = \frac{1}{2m} P_{ij} \pi^i \pi^j \tag{1.8}$$

where $P_{ij} = \delta_{ij} - \hat{d}_i \hat{d}_j$ is the projector transverse to the dipole moment, and $\hat{\boldsymbol{d}}$ is the unit vector in the direction of the dipole moment. This system has the expected planon symmetries and (im)mobilty, which can be seen as a consequence of a novel mixed Carroll-Galilei symmetry. In the transverse direction, the particle admits Galilei boosts and can therefore freely move. In the longitudinal direction, it has a Carroll boost symmetry and is therefore stuck, since isolated Carroll particles [19, 20] like fractons [21] cannot move.

Following [22], this particle action can be consistently coupled to the traceless scalar gauge theory [23]

$$S_{\text{int}}[\boldsymbol{z}, \boldsymbol{d}] = \int dt d^3x \left[\phi\rho - A_{ij} J^{ij}\right] = \int dt \left[\partial_i \phi(t, \boldsymbol{z}(t)) d^i + A_{ij}(t, \boldsymbol{z}(t)) d^{\langle i} \dot{z}^{j\rangle}\right]\,. \tag{1.9}$$

where $\phi$ represents the scalar potential, while $A_{ij}$ is the symmetric and traceless tensor potential. Here and in the following, angle brackets will implement the symmetric and traceless projection. The variation of the coupled system leads to the generalised Lorentz force law presented by Pretko [23], but now follows from an action principle including a planon particle.

In section 4.5, we discuss the quantisation of the composite model and obtain a gaussian Schrödinger-like theory described by the field $\psi(t, \boldsymbol{x})$, with action

$$S[\psi, \psi^*] = \int dt d^3x \left[ i\psi^* \mathcal{D}_t \psi - \frac{1}{2m} P^{ij} \mathcal{D}_i \psi (\mathcal{D}_j \psi)^* \right], \tag{1.10}$$

where we have introduced $\mathcal{D}_t = \partial_t - i d^i \partial_i \phi$ and $\mathcal{D}_i = i\partial_i + A_{ik} d^k$ with $\boldsymbol{d}$ being a constant parameter. This model already appeared in [24, 25] (see also [7, II.B.3]) in the context of fracton/elasticity duality. The uncoupled theory once again exhibits the Carroll-Galilei symmetry previously discussed (see section 4.6).

In section 5 we study the planon elementary quantum systems, i.e., the unitary irreducible representations (UIRs) of the centrally extended planon group. We use the fact that the centrally extended planon group can be written as a semidirect product $B \ltimes A$ with $B$ isomorphic to the Bargmann group and $A$ a two-dimensional abelian group, in order to apply the Mackey method to construct UIRs. As shown in appendix C, the semidirect product is regular and hence Mackey theory guarantees that all UIRs can be obtained as induced representations. There are two kinds of UIRs: those which are induced from UIRs of the Bargmann group and those which are induced from UIRs of a Carroll subgroup of the Bargmann group. This allows us to borrow (after some translation) the results of [1] on the UIRs of the Bargmann group and of [26] on the UIRs of the Carroll group in order to determine the UIRs of the centrally extended planon group. Rawnsley's theorem [27] guarantees that these UIRs are obtained by geometric quantisation of coadjoint orbits and we display this correspondence in table 3.

We close with a discussion in section 6, where we mention various generalisations and also argue that defects in crystals at low temperatures are carrollian.

The paper contains three appendices. In appendix A we summarise the results in [1] about the Bargmann group in planon language, setting up a useful dictionary between the two. Appendix B, contains a technical result about (not necessarily universal) central extensions. Finally, appendix C checks that the semidirect product description of the centrally extended planon group is regular, a technical result needed to guarantee that the Mackey method is exhaustive.

## 2 Classical planon particles

We follow the philosophy of Souriau [28], in which classical planon particles are to be identified with homogeneous symplectic manifolds of the planon group. Unlike the case of fractons, which we treated in this language in [21, 26], the planon group has nontrivial symplectic cohomology and that means that its homogeneous symplectic manifolds are not necessarily coadjoint orbits of the planon group, but of a one-dimensional central extension, which we will describe in section 2.1. After describing this group we discuss its coadjoint orbits in general dimension in section 2.2, before specialising to spatial dimension $n = 3$ in section 2.3. We list all the coadjoint orbits explicitly as zero loci of equations and give a representative for each orbit. These are listed, along with the stabiliser of the chosen representative, in table 1. This will then be the departing point to analyse the elementary particle dynamics and their mobility in section 3.

### 2.1 The central extension of the planon group

Let $G_{\text{pla}} \cong B \times \mathbb{R}$ denote the (connected, simply-connected) planon group, which is isomorphic to the product of the (connected, simply-connected) Bargmann group $B$ and the additive group of reals. The Lie algebra $\mathfrak{g}_{\text{pla}}$ is the span of $\langle L_{ab}, P_a, D_a, Z, Q, H\rangle$ where the Bargmann algebra $\mathfrak{b}$ in planon language is the span of $\mathfrak{b} = \langle L_{ab}, D_a, P_a, Z, Q\rangle$ with $H$ spanning the Lie algebra of $\mathbb{R}$. In the language of our recent summary [1] of Galilei particles, what we call $P_a, D_a, Z, Q$ here are there called $B_a, P_a, H, M$, respectively. We will be reusing many of the results in [1], but translating the generators into planon language. The precise dictionary is the subject of appendix A. The Lie brackets of $\mathfrak{g}_{\text{pla}}$ are given by the standard kinematical Lie brackets, which say that $\mathfrak{r} = \langle L_{ab}\rangle \cong \mathfrak{so}(n)$ is the rotational subalgebra, $D_a, P_a$ are vectors and $Z, Q, H$ are scalars, and in addition the following nonzero brackets:[2]

$$[P_a, Z] = D_a \qquad \text{and} \qquad [P_a, D_b] = \delta_{ab} Q. \tag{2.1}$$

The canonical dual basis for $\mathfrak{g}^*_{\text{pla}}$ is $\left\langle \lambda^{ab}, \pi^a, \delta^a, \zeta, \theta, \eta \right\rangle$. Central extensions are classified up to equivalence[3] by the Chevalley-Eilenberg group $H^2(\mathfrak{g}_{\text{pla}}; \mathbb{R})$. Provided that $n \geq 3$, which we will assume and in fact later specialise to $n = 3$, we can compute this group from the $\mathfrak{r}$-basic subcomplex $C^\bullet(\mathfrak{g}_{\text{pla}}, \mathfrak{r}; \mathbb{R})$ consisting of $\mathfrak{r}$-invariant cochains which have no legs along $\mathfrak{r}$. This complex is very easy to describe:

$$\begin{aligned} C^1(\mathfrak{g}_{\text{pla}}, \mathfrak{r}; \mathbb{R}) &= \langle \eta, \theta, \zeta\rangle \\ C^2(\mathfrak{g}_{\text{pla}}, \mathfrak{r}; \mathbb{R}) &= \left\langle \delta_{ab}\pi^a \wedge \delta^b, \eta\wedge\theta, \eta\wedge\zeta, \theta\wedge\zeta \right\rangle. \end{aligned} \tag{2.2}$$

The Chevalley-Eilenberg differential $\partial : C^1(\mathfrak{g}_{\text{pla}}, \mathfrak{r}; \mathbb{R}) \to C^2(\mathfrak{g}_{\text{pla}}, \mathfrak{r}; \mathbb{R})$ is given by

$$\partial\eta = \partial\zeta = 0 \qquad \text{and} \qquad \partial\theta = \delta_{ab}\pi^a \wedge \delta^b. \tag{2.3}$$

We do not need to know what $\partial\pi^a$ is for this calculation. This is because the only generator in $C^2(\mathfrak{g}_{\text{pla}}, \mathfrak{r}; \mathbb{R})$ which involves $\pi^a$ is obviously a coboundary. The 2-cocycles are spanned by $\left\langle \delta_{ab}\pi^a \wedge \delta^b, \eta\wedge\zeta \right\rangle$ and hence the cohomology is one-dimensional:

$$H^2(\mathfrak{g}_{\text{pla}}; \mathbb{R}) \cong H^2(\mathfrak{g}_{\text{pla}}, \mathfrak{r}; \mathbb{R}) = \langle [\eta\wedge\zeta]\rangle \tag{2.4}$$

spanned by the cohomology class of $\eta \wedge \zeta$. This signals the existence of a nontrivial central extension $\mathfrak{g}$ of the planon algebra $\mathfrak{g}_{\text{pla}}$ obtained by adding a new generator $W$ and a new Lie bracket

$$[H, Z] = W. \tag{2.5}$$

[2]To highlight the translation, these are the Bargmann brackets $[B_a, H] = P_a$ and $[B_a, P_b] = \delta_{ab} M$.

[3]Equivalence of central extensions refines the notion of Lie algebra isomorphism: two central extensions might be isomorphic as Lie algebras and yet inequivalent as central extensions, since equivalence requires that the isomorphism be the identity both on the Lie algebra being extended and on the central ideal.

In summary, we will let $G$ denote the (connected, simply-connected) centrally extended planon group, with Lie algebra $\mathfrak{g} = \langle L_{ab}, P_a, D_a, Q, Z, H, W\rangle$ and Lie brackets

$$[P_a, Z] = D_a, \qquad [P_a, D_b] = \delta_{ab} Q \qquad \text{and} \qquad [H, Z] = W, \tag{2.6}$$

beyond the kinematical ones involving $L_{ab}$. In the nomenclature of [29], this is a $(4,2)$ kinematical Lie algebra (KLA).

### 2.2 Coadjoint orbits

We now work out the coadjoint orbits of the centrally extended planon group $G$. We observe that $G \cong B \ltimes A$, where $A$ is the abelian group with Lie algebra $\mathfrak{a} = \langle H, W\rangle$ and $B$ is the Bargmann group with Lie algebra $\mathfrak{b} = \langle L_{ab}, P_a, D_a, Q, Z\rangle$ as already described above. In [1] we worked out the coadjoint orbits of the Bargmann group and we can re-use these results thanks to the following lemma. The notation $\operatorname{ann}\mathfrak{a}$ in the Lemma means the annihilator of $\mathfrak{a} \subset \mathfrak{g}$. This is the subspace of $\mathfrak{g}^*$ consisting of all linear functions on $\mathfrak{g}$ which vanish identically on $\mathfrak{a}$. Two of the basic isomorphism theorems in linear algebra are that $\mathfrak{a}^* \cong \mathfrak{g}^*/\operatorname{ann}\mathfrak{a}$ and dually that $\operatorname{ann}\mathfrak{a} \cong (\mathfrak{g}/\mathfrak{a})^*$.

**Lemma 1.** *Let $G = B \ltimes A$ where $A$ is abelian and let $\beta \in \operatorname{ann}\mathfrak{a} \subset \mathfrak{g}^*$ be a moment of $B$. Then for all $a \in A$, $\operatorname{Ad}^*_a \beta = \beta$.*

*Proof.* Let $a \in A$ and $\beta \in \operatorname{ann}\mathfrak{a}$. Then $\operatorname{Ad}^*_a \beta = \beta \circ \operatorname{Ad}_{a^{-1}}$. Let $X \in \mathfrak{g}$.

We claim that $\operatorname{Ad}_{a^{-1}} X = X \mod \mathfrak{a}$. Indeed, using that $A$ is a normal subgroup

$$\begin{aligned}\exp(t \operatorname{Ad}_{a^{-1}} X) &= a^{-1} \exp(tX) a \\ &= a^{-1} \exp(tX) a \exp(-tX) \exp(tX) \\ &= a^{-1} a(t) \exp(tX),\end{aligned}$$

where $a(t)$ is a curve in $A$ through $a$. Differentiating with respect to $t$ at $t = 0$,

$$\operatorname{Ad}_{a^{-1}} X = X + \underbrace{a^{-1}\dot{a}(0)}_{\in \mathfrak{a}}.$$

Therefore,

$$\begin{aligned}\langle \operatorname{Ad}^*_a \beta, X\rangle &= \langle \beta, \operatorname{Ad}_{a^{-1}} X\rangle \\ &= \langle \beta, X \mod \mathfrak{a}\rangle && \text{(by the above claim)} \\ &= \langle \beta, X\rangle && (\text{since } \beta \in \operatorname{ann}\mathfrak{a})\end{aligned}$$

and since this holds for all $X \in \mathfrak{g}$, it follows that $\operatorname{Ad}^*_a \beta = \beta$. □

This has as a consequence that if $\beta$ is a Bargmann moment, its coadjoint orbit under $G$ agrees with its Bargmann coadjoint orbit. Indeed, we can write every $g \in G$ uniquely as $g = ba$ with $b \in B$ and $a \in A$, and hence, since $\operatorname{Ad}^*$ is a representation,

$$\operatorname{Ad}^*_g \beta = \operatorname{Ad}^*_b \operatorname{Ad}^*_a \beta = \operatorname{Ad}^*_b \beta, \tag{2.7}$$

where we have used the Lemma in the second equality.

If $\tau = \alpha + \beta \in \mathfrak{g}^*$ is such that $\beta \in \operatorname{ann}\mathfrak{a} \cong \mathfrak{b}^*$ and $\alpha \in \operatorname{ann}\mathfrak{b} \cong \mathfrak{a}^*$, then under $g = ba$, we have that

$$\begin{aligned} \operatorname{Ad}_g^*(\alpha+\beta) &= \operatorname{Ad}_b^* \operatorname{Ad}_a^*(\alpha+\beta) \\ &= \operatorname{Ad}_b^* \beta + \operatorname{Ad}_b^* \operatorname{Ad}_a^* \alpha, \end{aligned} \tag{2.8}$$

where $\operatorname{Ad}_b^* \beta$ can be read off from the results in [1], suitably translated, as in appendix A. See in particular equation (A.4) in general and equation (A.5) for $n = 3$.

Let us use these results to calculate the coadjoint action of the generic element of $G$, denoted

$$\mathbf{g}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R, s, w) = \mathbf{b}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R) \exp(sH + wW), \tag{2.9}$$

on the moment

$$\mathbf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, J, E, c) = \frac{1}{2} J_{ab}\lambda^{ab} + p_a \pi^a - d_a \delta^a + \tilde{q}\zeta + q\theta + E\eta + c\omega, \tag{2.10}$$

where we have introduced $\eta, \omega \in \mathfrak{g}^*$ obeying $\langle \omega, W\rangle = 1$ and $\langle \eta, H\rangle = 1$ and zero otherwise. We calculate that $\operatorname{ad}_Z^* \omega = \eta$ and $\operatorname{ad}_H^* \omega = -\zeta$. The generic $G$ moment is a sum $\beta + E\eta + c\omega$, where $\beta$ is a Bargmann moment.

Let $a = \exp(sH + wW)$ and let us calculate $\operatorname{Ad}_a^*(E\eta + cW)$. Since $W$ is central, $\operatorname{ad}_W^* = 0$ and hence $\operatorname{Ad}_a^* = \exp(s \operatorname{ad}_H^*)$, resulting in

$$\operatorname{Ad}_a^*(E\eta + cW) = \exp(s \operatorname{ad}_H^*)(E\eta + cW) = E\eta + c\omega - cs\zeta. \tag{2.11}$$

Let $b = \mathbf{b}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R)$ and let us calculate $\operatorname{Ad}_b^*(E\eta + c\omega - cs\zeta)$. Only $Z$ acts nontrivially on this subspace, hence

$$\operatorname{Ad}_b^*(E\eta + c\omega - cs\zeta) = \exp(\tilde{\varphi} \operatorname{ad}_Z^*)(E\eta + c\omega - cs\zeta) = (E + c\tilde{\varphi})\eta + c\omega - cs\zeta. \tag{2.12}$$

Putting this all together we now have that

$$\operatorname{Ad}_{\mathbf{g}(\varphi,\tilde{\varphi},\boldsymbol{\beta},\boldsymbol{a},R,s,w)}^* \mathbf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, J, E, c) = \mathbf{M}(q', \tilde{q}', \boldsymbol{d}', \boldsymbol{p}', J', E', c') \tag{2.13}$$

where, using equation (A.4),

$$\begin{aligned} J' &= RJR^T + \boldsymbol{\beta}(R\boldsymbol{d} + q\boldsymbol{a})^T - (R\boldsymbol{d} + q\boldsymbol{a})\boldsymbol{\beta}^T + (R\boldsymbol{p})\boldsymbol{a}^T - \boldsymbol{a}(R\boldsymbol{p})^T \\ \boldsymbol{p}' &= R\boldsymbol{p} + q\boldsymbol{\beta} + \tilde{\varphi}(R\boldsymbol{d} + q\boldsymbol{a}) \\ \boldsymbol{d}' &= R\boldsymbol{d} + q\boldsymbol{a} \\ \tilde{q}' &= \tilde{q} + R\boldsymbol{d}\cdot\boldsymbol{a} + \frac{1}{2}q\|\boldsymbol{d}\|^2 - cs \\ q' &= q \\ c' &= c \\ E' &= E + c\tilde{\varphi}, \end{aligned} \tag{2.14}$$

and in the special case of three dimensions ($n = 3$) we have, using equation (A.5), that

$$\begin{aligned}
\boldsymbol{j}' &= R\boldsymbol{j} - \boldsymbol{\beta} \times (R\boldsymbol{d} + q\boldsymbol{a}) + \boldsymbol{a} \times R\boldsymbol{p} \\
\boldsymbol{p}' &= R\boldsymbol{p} + q\boldsymbol{\beta} + \tilde{\varphi}(R\boldsymbol{d} + q\boldsymbol{a}) \\
\boldsymbol{d}' &= R\boldsymbol{d} + q\boldsymbol{a} \\
\tilde{q}' &= \tilde{q} + R\boldsymbol{d} \cdot \boldsymbol{a} + \frac{1}{2}q\|\boldsymbol{d}\|^2 - cs \\
q' &= q \\
c' &= c \\
E' &= E + c\tilde{\varphi}.
\end{aligned} \tag{2.15}$$

### 2.3 Coadjoint orbits for $n = 3$

Since $Q$ and $W$ span the centre of $\mathfrak{g}$, they span the kernel of the (co)adjoint representation of $\mathfrak{g}$. This means that the group acting effectively on $\mathfrak{g}^*$ is the quotient of $G$ by the centre, which is the subgroup generated by $Q$ and $W$. This quotient is isomorphic to the direct product of the Galilei group with the one-parameter subgroup generated by $H$. In other words, coadjoint orbits are homogeneous symplectic manifolds of Gal $\times \mathbb{R}$, where Gal the group generated by $\left\langle \overline{L}_{ab}, \overline{P}_a, \overline{D}_a, \overline{Z} \right\rangle$, where $X \mapsto \overline{X}$ is the quotient of $\mathfrak{g}$ by the centre. Homogeneous symplectic manifolds of Gal $\times \mathbb{R}$ are coadjoint orbits of a *one-dimensional* central extension. The group Gal $\times \mathbb{R}$ has two-dimensional symplectic cohomology and the group $G$ is a two-dimensional central extension, which although not universal (the planon group is not perfect and therefore has no universal central extension), nevertheless is such that it surjects onto any one-dimensional central extension. We can see this at the level of the Lie algebra and the details are in appendix B. Hence every one-dimensional central extension of Gal $\times \mathbb{R}$ is the quotient of $G$ by a one-dimensional central subgroup. Such a subgroup is generated by a line in the plane in $\mathfrak{g}$ spanned by $Q$ and $W$. There are two classes of such lines: those spanned by $W$ and those spanned by $Q - \lambda W$, with $\lambda \in \mathbb{R}$.

Quotienting by the one-parameter subgroup generated by $W$ we obtain the planon group $B \times \mathbb{R}$, with $B$ the Bargmann group. These are the coadjoint orbits with $c = 0$, which are in one-to-one correspondence with the Bargmann coadjoint orbits, recalled in table 5, embedded in $\mathfrak{g}^*$ by placing them at a fixed value of $E$.

The orbits with $c \neq 0$ are coadjoint orbits of the quotient of $G$ by the central subgroup generated by $Q - \lambda W$. Those coadjoint orbits sit inside the hyperplane of $\mathfrak{g}^*$ corresponding to the zeros of the linear function $Q - \lambda W$, hence those moments with $q = \lambda c$. The Lie algebra of this group is generated by $\langle L_{ab}, P_a, D_a, H, Z, W \rangle$ and the non-generic brackets are now

$$[P_a, Z] = D_a, \qquad [P_a, D_b] = \delta_{ab}\lambda W \qquad \text{and} \qquad [H, Z] = W. \tag{2.16}$$

We have two cases to consider: $\lambda = 0$ and $\lambda \neq 0$, in which case we may set $\lambda = 1$ by rescaling $W$ and $H$. (Later in section 5, when we discuss UIRs, it is more convenient not to do the rescaling and keep $\lambda$ general.)

The case $\lambda = 0$ is when we quotient by the central subgroup generated by $Q$ and hence corresponds to those coadjoint orbits with $q = 0$ and $c \neq 0$. From equation (2.15) we see that the coadjoint action of Gal $\times\ \mathbb{R}$ is given in this case by

$$\begin{aligned}
\boldsymbol{j}' &= R\boldsymbol{j} - \boldsymbol{\beta} \times R\boldsymbol{d} + \boldsymbol{a} \times R\boldsymbol{p} \\
\boldsymbol{p}' &= R\boldsymbol{p} + \tilde{\varphi} R\boldsymbol{d} \\
\boldsymbol{d}' &= R\boldsymbol{d} \\
\tilde{q}' &= \tilde{q} + R\boldsymbol{d} \cdot \boldsymbol{a} - cs \\
E' &= E + c\tilde{\varphi}.
\end{aligned} \tag{2.17}$$

In this case it is evident that $\|\boldsymbol{d}\|^2$, $\|\boldsymbol{p} \times \boldsymbol{d}\|^2$ and $E\|\boldsymbol{d}\|^2 - c\boldsymbol{p} \cdot \boldsymbol{d}$ are polynomial functions which are constant on the orbits. We may enumerate these orbits as follows.

- ($\boldsymbol{d} = \boldsymbol{0}$) In this case, the coadjoint action simplifies to

$$\begin{aligned}
\boldsymbol{j}' &= R\boldsymbol{j} + \boldsymbol{a} \times R\boldsymbol{p} \\
\boldsymbol{p}' &= R\boldsymbol{p} \\
\tilde{q}' &= \tilde{q} - cs \\
E' &= E + c\tilde{\varphi},
\end{aligned} \tag{2.18}$$

  from where we observe that $\|\boldsymbol{p}\|^2$ and $\boldsymbol{p} \cdot \boldsymbol{j}$ are polynomial invariants of the orbits.

  – ($\boldsymbol{p} = \boldsymbol{0}$) In this case the coadjoint action simplifies further to

$$\begin{aligned}
\boldsymbol{j}' &= R\boldsymbol{j} \\
\tilde{q}' &= \tilde{q} - cs \\
E' &= E + c\tilde{\varphi},
\end{aligned} \tag{2.19}$$

  and hence $\|\boldsymbol{j}\|^2$ is invariant.
  If $\boldsymbol{j} = \boldsymbol{0}$, then the orbit is the affine plane where $E, \tilde{\varphi}$ can take any values and $c = c_0 \neq 0$. The equations for these two-dimensional orbits are

$$\boldsymbol{p} = \boldsymbol{d} = \boldsymbol{j} = \boldsymbol{0}, \qquad q = 0 \qquad \text{and} \qquad c = c_0 \neq 0. \tag{2.20}$$

  We see that there are 11 equations, as expected for two-dimensional orbits of 13-dimensional Lie group.
  If $\|\boldsymbol{j}\|^2 = \ell^2 > 0$, the orbit is the product of a sphere of radius $\ell$ and the affine plane above. The orbits are four-dimensional and indeed they are cut out by 9 polynomial equations:

$$\boldsymbol{p} = \boldsymbol{d} = \boldsymbol{0}, \qquad \|\boldsymbol{j}\|^2 = \ell^2, \qquad q = 0 \qquad \text{and} \qquad c = c_0 \neq 0. \tag{2.21}$$

  – ($\boldsymbol{p} \neq \boldsymbol{0}$) Let $\|\boldsymbol{p}\|^2 = p^2 > 0$ and $\boldsymbol{p} \cdot \boldsymbol{j} = hp \in \mathbb{R}$. These two equations are supplemented by 5 more equations: $\boldsymbol{d} = \boldsymbol{0}$, $q = 0$ and $c = c_0 \neq 0$. The orbit is 6-dimensional and looks like the product of the cotangent bundle of the 2-sphere of radius $p$ and the affine plane above. A convenient orbit representative for this orbit is $\mathsf{M}(0, 0, \boldsymbol{0}, p\boldsymbol{u}, h\boldsymbol{u}, 0, c_0)$ where $\boldsymbol{u}$ is a fixed unit vector. It is not hard to determine that the stabiliser of that point is 7-dimensional, as expected.

- ($\boldsymbol{d} \neq \boldsymbol{0}$) Let $\|\boldsymbol{d}\|^2 = d^2 > 0$.
  - ($\boldsymbol{p} \times \boldsymbol{d} = \boldsymbol{0}$) This says that $\boldsymbol{p}$ and $\boldsymbol{d}$ are collinear: $\boldsymbol{p} = \pm\frac{p}{d}\boldsymbol{d}$, with the sign denoting whether they are parallel or antiparallel. In this case, $\boldsymbol{j} \cdot \boldsymbol{d}$ is now invariant and so is $Ed \mp cp$. The coadjoint action becomes now

$$\begin{aligned}
\boldsymbol{j}' &= R\boldsymbol{j} + \left(\pm\frac{p}{d}\boldsymbol{a} - \boldsymbol{\beta}\right) \times R\boldsymbol{d} \\
\boldsymbol{p}' &= \left(\pm\frac{p}{d} + \tilde{\varphi}\right) R\boldsymbol{d} \\
\boldsymbol{d}' &= R\boldsymbol{d} \\
\tilde{q}' &= \tilde{q} + R\boldsymbol{d} \cdot \boldsymbol{a} - cs \\
E' &= E + c\tilde{\varphi}.
\end{aligned} \tag{2.22}$$

  The orbits are six-dimensional, being cut out by 7 equations:

$$\begin{gathered}
q = 0, \quad c = c_0 \neq 0, \quad \boldsymbol{p} \times \boldsymbol{d} = \boldsymbol{0}, \quad \|\boldsymbol{d}\|^2 = d^2 > 0, \\
\|E\boldsymbol{d} - c\boldsymbol{p}\|^2 = E_0^2 d^2 \geq 0 \quad \text{and} \quad \boldsymbol{j} \cdot \boldsymbol{d} = hd \in \mathbb{R}.
\end{gathered} \tag{2.23}$$

  We can take as orbit representative the moment $\mathsf{M}(0, 0, d\boldsymbol{u}, \pm\frac{E_0 d}{c_0}\boldsymbol{u}, h\boldsymbol{u}, 0, c_0)$, where $\boldsymbol{u}$ is a fixed unit vector. It is not hard to show that the stabiliser subgroup of that moment is 7-dimensional, as expected. We must distinguish a special case of this orbit: when $E_0 = 0$. In that case, we can choose as representative the moment with both $E = 0$ and $\boldsymbol{p} = \boldsymbol{0}$: $\mathsf{M}(0, 0, d\boldsymbol{u}, \boldsymbol{0}, h\boldsymbol{u}, 0, c_0)$. There is no longer a sign ambiguity in this case.
  - ($\boldsymbol{p} \times \boldsymbol{d} \neq \boldsymbol{0}$) We decompose $\boldsymbol{p} = \frac{p}{d}\cos\theta \boldsymbol{d} + \boldsymbol{p}_\perp$, where $\boldsymbol{p}_\perp \cdot \boldsymbol{d} = 0$. Let $\|\boldsymbol{p}_\perp\|^2 = p^2 \sin^2\theta$. Then the orbit is cut out by 5 equations:

$$\begin{aligned}
q = 0, \quad c = c_0 \neq 0, \quad \|\boldsymbol{d}\|^2 = d^2 > 0, \quad \|\boldsymbol{p} \times \boldsymbol{d}\|^2 = (pd\sin\theta)^2 > 0 \\
\text{and} \quad Ed - c_0 p \cos\theta = \epsilon_0 d \in \mathbb{R}.
\end{aligned} \tag{2.24}$$

  The orbit is parametrised by the values $d, p, \theta, c$, where the invariants are $c, d, p\sin\theta$ and $\epsilon_0 := E - \frac{cp}{d}\cos\theta$. We can take as orbit representative the moment $\mathsf{M}(0, 0, d\boldsymbol{u}, p\sin\theta\boldsymbol{u}^\perp - \frac{\epsilon_0 d}{c_0}\boldsymbol{u}, \boldsymbol{0}, 0, c_0)$, where $\boldsymbol{u}, \boldsymbol{u}^\perp$ are chosen perpendicular unit vectors. It is not hard to determine the stabiliser subgroup and show that it is 5-dimensional, as expected.

The case $\lambda = 1$ sets $q = c$ in equation (2.15) to give

$$\begin{aligned}
\boldsymbol{j}' &= R\boldsymbol{j} - \boldsymbol{\beta} \times (R\boldsymbol{d} + c\boldsymbol{a}) + \boldsymbol{a} \times R\boldsymbol{p} \\
\boldsymbol{p}' &= R\boldsymbol{p} + c\boldsymbol{\beta} + \tilde{\varphi}(R\boldsymbol{d} + c\boldsymbol{a}) \\
\boldsymbol{d}' &= R\boldsymbol{d} + c\boldsymbol{a} \\
\tilde{q}' &= \tilde{q} + R\boldsymbol{d} \cdot \boldsymbol{a} + \frac{1}{2}c\|\boldsymbol{d}\|^2 - cs \\
E' &= E + c\tilde{\varphi}.
\end{aligned} \tag{2.25}$$



JHEP11(2025)030

| # | Orbit representative $\alpha = \mathbf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{j}, E, c)$ | Stabiliser $G_\alpha \ni \mathbf{g}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R, s, w)$ | Equations for orbits |
|---|---|---|---|
| 0 | $\mathbf{M}(0, \tilde{q}_0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, E_0, 0)$ | $G$ | $c = 0, E = E_0, q = 0, \tilde{q} = \tilde{q}_0, \boldsymbol{d} = \boldsymbol{p} = \boldsymbol{j} = \boldsymbol{0}$ |
| 2 | $\mathbf{M}(0, \tilde{q}_0, \boldsymbol{0}, \boldsymbol{0}, \ell\boldsymbol{u}, E_0, 0)$ | $\{\mathbf{g}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R, s, w) \mid R\boldsymbol{u} = \boldsymbol{u}\}$ | $c = 0, E = E_0, q = 0, \tilde{q} = \tilde{q}_0, \boldsymbol{d} = \boldsymbol{p} = \boldsymbol{0}, \lVert\boldsymbol{j}\rVert = \ell$ |
| $2'$ | $\mathbf{M}(0, 0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, 0, c_0)$ | $\{\mathbf{g}(\varphi, 0, \boldsymbol{\beta}, \boldsymbol{a}, R, 0, w)\}$ | $c = c_0, q = 0, \boldsymbol{p} = \boldsymbol{d} = \boldsymbol{j} = \boldsymbol{0}$ |
| 4 | $\mathbf{M}(0, \tilde{q}_0, \boldsymbol{0}, p\boldsymbol{u}, h\boldsymbol{u}, E_0, 0)$ | $\{\mathbf{g}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, a\boldsymbol{u}, R, s, w) \mid R\boldsymbol{u} = \boldsymbol{u}\}$ | $c = 0, E = E_0, q = 0, \tilde{q} = \tilde{q}_0, \boldsymbol{d} = \boldsymbol{0}, \lVert\boldsymbol{p}\rVert = p, \boldsymbol{j} \cdot \boldsymbol{p} = hp$ |
| $4'$ | $\mathbf{M}(0, 0, \boldsymbol{0}, \boldsymbol{0}, \ell\boldsymbol{u}, 0, c_0)$ | $\{\mathbf{g}\varphi, 0, \boldsymbol{\beta}, \boldsymbol{a}, R, 0, w) \mid R\boldsymbol{u} = \boldsymbol{u}\}$ | $c = c_0, q = 0, \boldsymbol{p} = \boldsymbol{d} = \boldsymbol{0}, \lVert\boldsymbol{j}\rVert = \ell$ |
| 6 | $\mathbf{M}(q_0, \tilde{q}_0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, E_0, 0)$ | $\{\mathbf{g}(\varphi, \tilde{\varphi}, \boldsymbol{0}, \boldsymbol{0}, R, s, w)\}$ | $c = 0, E = E_0, q = q_0, \frac{1}{2q}(\lVert\boldsymbol{d}\rVert^2 - 2q\tilde{q}) = \tilde{q}_0, q\boldsymbol{j} = \boldsymbol{d} \times \boldsymbol{p}$ |
| $6'$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, \boldsymbol{0}, \boldsymbol{0}, E_0, 0)$ | $\{\mathbf{g}(\varphi, 0, \beta\boldsymbol{u}, \boldsymbol{a}, R, s, w) \mid R\boldsymbol{u} = \boldsymbol{u}, \boldsymbol{a} \cdot \boldsymbol{u} = 0\}$ | $c = 0, E = E_0, q = 0, \lVert\boldsymbol{d}\rVert = d > 0, \boldsymbol{d} \times \boldsymbol{p} = \boldsymbol{0}$ |
| $6''$ | $\mathbf{M}(0, 0, \boldsymbol{0}, p\boldsymbol{u}, h\boldsymbol{u}, 0, c_0)$ | $\{\mathbf{g}(\varphi, 0, \boldsymbol{\beta}, a\boldsymbol{u}, R, 0, w) \mid R\boldsymbol{u} = \boldsymbol{u}\}$ | $c = c_0, q = 0, \boldsymbol{d} = \boldsymbol{0}, \lVert\boldsymbol{p}\rVert = p, \boldsymbol{j} \cdot \boldsymbol{p} = hp$ |
| $6'''_\pm$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, \pm\frac{E_0 d}{c_0}\boldsymbol{u}, h\boldsymbol{u}, 0, c_0)$ | $\{\mathbf{g}(\varphi, 0, \boldsymbol{\beta}, \boldsymbol{a}, R, 0, w) \mid R\boldsymbol{u} = \boldsymbol{u}, (d\boldsymbol{\beta} \mp p\boldsymbol{a}) \times \boldsymbol{u} = \boldsymbol{0}\}$ | $c = c_0, q = 0, \boldsymbol{p} \times \boldsymbol{d} = \boldsymbol{0}, \lVert\boldsymbol{d}\rVert = d, \boldsymbol{j} \cdot \boldsymbol{d} = hd, \lVert E\boldsymbol{d} - c\boldsymbol{p}\rVert = E_0 d$ |
| $6'''_0$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, \boldsymbol{0}, h\boldsymbol{u}, 0, c_0)$ | $\{\mathbf{g}(\varphi, 0, \boldsymbol{\beta}, \boldsymbol{a}, R, 0, w) \mid R\boldsymbol{u} = \boldsymbol{u}, d\boldsymbol{\beta} \times \boldsymbol{u} = \boldsymbol{0}\}$ | $c = c_0, q = 0, \boldsymbol{p} = \frac{E}{c}\boldsymbol{d}, \lVert\boldsymbol{d}\rVert = d, \boldsymbol{j} \cdot \boldsymbol{d} = hd$ |
| 8 | $\mathbf{M}(q_0, \tilde{q}_0, \boldsymbol{0}, \boldsymbol{0}, \ell\boldsymbol{u}, E_0, 0)$ | $\{\mathbf{g}(\varphi, \tilde{\varphi}, \boldsymbol{0}, \boldsymbol{0}, R, s, w) \mid R\boldsymbol{u} = \boldsymbol{u}\}$ | $c = 0, E = E_0, q = q_0, \frac{1}{2q}(\lVert\boldsymbol{d}\rVert^2 - 2q\tilde{q}) = \tilde{q}_0, \lVert q\boldsymbol{j} - \boldsymbol{d} \times \boldsymbol{p}\rVert = \ell$ |
| $8'$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, p\boldsymbol{u}^\perp, \boldsymbol{0}, E_0, 0)$ | $\{\mathbf{g}(\varphi, 0, \beta\boldsymbol{u}, a\boldsymbol{u}^\perp, \pm I, s, w)\}$ | $c = 0, E = E_0, q = 0, \lVert\boldsymbol{d}\rVert = d, \lVert\boldsymbol{d} \times \boldsymbol{p}\rVert = dp$ |
| $8''$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, p\sin\theta\boldsymbol{u}^\perp - \frac{\epsilon_0 d}{c_0}\boldsymbol{u}, \boldsymbol{0}, 0, c_0)$ | $\{\mathbf{g}(\varphi, 0, \boldsymbol{\beta}, \boldsymbol{a}, \pm I, 0, w) \mid \boldsymbol{a} \times \boldsymbol{p} = \boldsymbol{\beta} \times \boldsymbol{d}\}$ | $c = c_0, q = 0, \lVert\boldsymbol{d}\rVert = d, \lVert\boldsymbol{p} \times \boldsymbol{d}\rVert = pd\sin\theta, (E - \epsilon_0)d^2 = c\boldsymbol{p} \cdot \boldsymbol{d}$ |
| $8'''$ | $\mathbf{M}(c_0, 0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, 0, c_0)$ | $\{\mathbf{g}(\varphi, 0, \boldsymbol{0}, \boldsymbol{0}, R, 0, w)\}$ | $c = q = c_0, c\boldsymbol{j} = \boldsymbol{d} \times \boldsymbol{p}$ |
| 10 | $\mathbf{M}(c_0, 0, \boldsymbol{0}, \boldsymbol{0}, \ell\boldsymbol{u}, 0, c_0)$ | $\{\mathbf{g}(\varphi, 0, \boldsymbol{0}, \boldsymbol{0}, R, 0, w) \mid R\boldsymbol{u} = \boldsymbol{u}\}$ | $c = q = c_0, \lVert\boldsymbol{j} - \frac{1}{c}\boldsymbol{d} \times \boldsymbol{p}\rVert = \ell$ |

**Table 1.** Coadjoint orbits of the (extended) planon group $G$. This table lists the different coadjoint orbits of the (extended) planon group ordered by increasing dimension. In each case we exhibit a label (from which one can read off the dimension), an orbit representative $\alpha \in \mathfrak{g}^*$, its stabiliser subgroup $G_\alpha$ inside the planon group and the equations defining the orbit. In this table, $\boldsymbol{u}$ and $\boldsymbol{u}^\perp$ stand for perpendicular unit vectors in $\mathbb{R}^3$. Wherever they appear, the parameters $c_0$ and $q_0$ are nonzero, whereas the parameters $\ell$, $d$, $p$ and $E_0$ are positive. The parameters $h$ and $\epsilon_0$ are arbitrary real numbers. The $\pm I$ in the stabiliser is due to the fact that we are working with the simply-connected spin group and $\pm I$ is the centre, which is in the kernel of the geometric action of the spin group on vectors via rotations.

We can set $s = \frac{1}{c}(\tilde{q} + R\boldsymbol{d} \cdot \boldsymbol{a}) + \frac{1}{2}\|\boldsymbol{d}\|^2$ and $\tilde{\varphi} = -E/c$ to set $\tilde{q}' = E' = 0$. Choose $\boldsymbol{a} = -\frac{1}{c}R\boldsymbol{d}$ to set $\boldsymbol{d}' = \boldsymbol{0}$ and choose $\boldsymbol{\beta} = -\frac{1}{c}R\boldsymbol{p}$ to set $\boldsymbol{p}' = 0$. Then we remain with $\boldsymbol{j}' = R(\boldsymbol{j} - \frac{1}{c}\boldsymbol{d} \times \boldsymbol{p})$. Therefore we have two kinds of orbits:

- those with $\boldsymbol{j}' = \boldsymbol{0}$, which are 8-dimensional orbits with stabiliser $\mathrm{SO}(3) \times \mathbb{R}^2$; and
- those with $\boldsymbol{j}' \neq \boldsymbol{0}$, which are 10-dimensional orbits with stabiliser the $\mathrm{SO}(2) \times \mathbb{R}^2$, with $\mathrm{SO}(2)$ the subgroup of $\mathrm{SO}(3)$ which preserves $\boldsymbol{j}'$.

The 8-dimensional orbits are determined by the equation $c\boldsymbol{j} - \boldsymbol{d} \times \boldsymbol{p} = \boldsymbol{0}$, apart from the ones which set the constant values of $q = c = c_0$. They consist of the points

$$\left\{\mathbf{M}(c_0, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, \frac{1}{q}\boldsymbol{d} \times \boldsymbol{p}, E, c_0) \,\middle|\, \boldsymbol{d}, \boldsymbol{p} \in \mathbb{R}^3 \quad \text{and} \quad \tilde{q}, E \in \mathbb{R}\right\} \subset \mathfrak{g}^*. \tag{2.26}$$

We may take $\mathbf{M}(c_0, 0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, 0, c_0)$ as orbit representative.

The 10-dimensional orbits are determined by the equations $\|c\boldsymbol{j} - \boldsymbol{d} \times \boldsymbol{p}\|^2 = \ell^2$, with $\ell > 0$ and $q = c = c_0$. They consist of the points

$$\left\{\mathbf{M}(c_0, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, \ell\boldsymbol{u}, E, c_0) \,\middle|\, \boldsymbol{d}, \boldsymbol{p} \in \mathbb{R}^3, \quad \boldsymbol{u} \in S^2 \subset \mathbb{R}^3 \quad \text{and} \quad \tilde{q}, E \in \mathbb{R}\right\} \subset \mathfrak{g}^*. \tag{2.27}$$

## 3 Elementary particles and mobility

In this section we study the dynamics of the elementary particles associated with the different coadjoint orbits, paying particular attention to their (restricted) mobility. Coadjoint orbits of the centrally extended planon group are homogeneous symplectic manifolds of the planon group. Let $\mathcal{O}_\alpha$ be the coadjoint orbit corresponding to $\alpha \in \mathfrak{g}^*$. The orbit map $\pi_\alpha : G \to \mathcal{O}_\alpha$ sends every $g \in G$ to $\mathrm{Ad}_g^* \alpha$. Let $\omega \in \Omega^2(\mathcal{O}_\alpha)$ denote the Kirillov-Kostant-Souriau symplectic form on the coadjoint orbit: it is a closed non-degenerate 2-form. Pulling back $\omega$ via the orbit map, we get a presymplectic form $\pi_\alpha^*\omega \in \Omega^2(G)$ in the group: a closed 2-form. In fact, a calculation shows that it is not just closed, but actually exact: $\pi_\alpha^*\omega = -d\langle\alpha, \vartheta\rangle$, where $\vartheta \in \Omega^1(G; \mathfrak{g})$ is the left-invariant Maurer-Cartan one-form on $G$ and $\langle -, -\rangle$ denotes the dual pairing. The primitive $\langle\alpha, \vartheta\rangle$ defines a variational problem for curves on $G$. If $\gamma : I \to G$, where $I$ is some interval containing 0 in the real line, then we define

$$S[\gamma] := \int_I \langle\alpha, \gamma^*\vartheta\rangle = \int_I \left\langle \alpha, \gamma(\tau)^{-1}\dot{\gamma}(\tau) \right\rangle d\tau, \tag{3.1}$$

where $\tau \in I$ is the parameter along the curve. In section 3.1 we calculate the left-invariant Maurer-Cartan one-form on $G$ and in section 3.2 we perform a preliminary geometric analysis of the particle trajectories resulting from the above variational principle. In section 3.3 we concentrate on some of the more interesting coadjoint orbits (those which are coadjoint orbits of the planon group itself) and discuss action functionals for trajectories on the corresponding aristotelian spacetime where the planons live.

### 3.1 Maurer-Cartan one-form

We record here the pull-back to the space of group parameters of the left-invariant Maurer-Cartan one-form on $G$. Let $g = ba$, with $b \in B$ and $a = \exp(tH + wW)$. Then

$$g^{-1}dg = a^{-1}(b^{-1}db)a + a^{-1}da = \exp(t\,\mathrm{ad}_H)(b^{-1}db) + dtH + dwW, \tag{3.2}$$

where we have used that $W$ is central. We can read off $b^{-1}db$ from equation (A.6):

$$b^{-1}db = \mathsf{A}\left(d\varphi - \boldsymbol{a}^T d\boldsymbol{\beta} - \frac{1}{2}\|\boldsymbol{a}\|^2 d\tilde{\varphi}, d\tilde{\varphi}, R^T(d\boldsymbol{\beta} + \boldsymbol{a}d\tilde{\varphi}), R^T d\boldsymbol{a}, R^T dR\right) \tag{3.3}$$

and hence

$$\exp(t\,\mathrm{ad}_H)(b^{-1}db) = \mathsf{A}\left(d\varphi - \boldsymbol{a}^T d\boldsymbol{\beta} - \frac{1}{2}\|\boldsymbol{a}\|^2 d\tilde{\varphi}, d\tilde{\varphi}, R^T(d\boldsymbol{\beta} + \boldsymbol{a}d\tilde{\varphi}), R^T d\boldsymbol{a}, R^T dR\right) + td\tilde{\varphi}W, \tag{3.4}$$

which we can put together to arrive at an expression for $g^{-1}dg$. Contracting that expression with the generic moment $\mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, J, E, c)$ we find (using equation (A.7)),

$$\begin{aligned}\left\langle \mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, J, E, c), g^{-1}dg \right\rangle &= c(dw + td\tilde{\varphi}) + Edt + qd\varphi - \left(\tilde{q} + \frac{1}{2}q\|\boldsymbol{a}\|^2 + (R\boldsymbol{\beta})^T\boldsymbol{a}\right) d\tilde{\varphi} \\ &\quad - (R\boldsymbol{d} + q\boldsymbol{a})^T d\boldsymbol{\beta} + (R\boldsymbol{p})^T d\boldsymbol{a} + \frac{1}{2}\operatorname{Tr} J^T R^T dR.\end{aligned} \tag{3.5}$$

Specialising to three dimensions, we find

$$\begin{aligned}\left\langle \mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{j}, E, c), g^{-1}dg \right\rangle &= c(dw + td\tilde{\varphi}) + Edt + qd\varphi - \left(\tilde{q} + \frac{1}{2}q\|\boldsymbol{a}\|^2 + R\boldsymbol{\beta}\cdot\boldsymbol{a}\right) d\tilde{\varphi} \\ &\quad - (R\boldsymbol{d} + q\boldsymbol{a})\cdot d\boldsymbol{\beta} + R\boldsymbol{p}\cdot d\boldsymbol{a} + \boldsymbol{j}\cdot\varepsilon^{-1}(R^{-1}dR).\end{aligned} \tag{3.6}$$



JHEP11(2025)030

| # | $\alpha = \mathbf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{j}, E, c) \in \mathfrak{g}^*$ | $\mathfrak{g}_\alpha$ | $\mathfrak{g}_\alpha \cap \mathfrak{g}_o$ | $\mathfrak{m}$ |
|---|---|---|---|---|
| $0$ | $\mathbf{M}(0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \mathbf{0}, E_0, 0)$ | $\mathfrak{g}$ | $\mathfrak{g}_o$ | $\langle H, P_a \rangle$ |
| $2$ | $\mathbf{M}(0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \ell \boldsymbol{u}, E_0, 0)$ | $\langle W, H, Q, Z, D_a, P_a, L_3 \rangle$ | $\langle W, Q, Z, D_a, L_3 \rangle$ | $\langle H, P_a \rangle$ |
| $2'$ | $\mathbf{M}(0, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, c_0)$ | $\langle W, Q, D_a, P_a, L_a \rangle$ | $\langle W, Q, D_a, L_a \rangle$ | $\langle P_a \rangle$ |
| $4$ | $\mathbf{M}(0, \tilde{q}_0, \mathbf{0}, p\boldsymbol{u}, h\boldsymbol{u}, E_0, 0)$ | $\langle W, H, Q, Z, D_a, P_3, L_3 \rangle$ | $\langle W, Q, Z, D_a, L_3 \rangle$ | $\langle H, P_3 \rangle$ |
| $4'$ | $\mathbf{M}(0, 0, \mathbf{0}, \mathbf{0}, \ell \boldsymbol{u}, 0, c_0)$ | $\langle W, Q, D_a, P_a, L_3 \rangle$ | $\langle W, Q, D_a, L_3 \rangle$ | $\langle P_a \rangle$ |
| $6$ | $\mathbf{M}(q_0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \mathbf{0}, E_0, 0)$ | $\langle W, H, Q, Z, L_a \rangle$ | $\langle W, Q, Z, L_a \rangle$ | $\langle H \rangle$ |
| $6'$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, \mathbf{0}, \mathbf{0}, E_0, 0)$ | $\langle W, H, Q, D_3, P_1, P_2, L_3 \rangle$ | $\langle W, Q, D_3, L_3 \rangle$ | $\langle H, P_1, P_2 \rangle$ |
| $6''$ | $\mathbf{M}(0, 0, \mathbf{0}, p\boldsymbol{u}, h\boldsymbol{u}, 0, c_0)$ | $\langle W, Q, D_a, P_3, L_3 \rangle$ | $\langle W, Q, D_a, L_3 \rangle$ | $\langle P_3 \rangle$ |
| $6'''_\pm$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, \pm\frac{E_0 d}{c_0}\boldsymbol{u}, h\boldsymbol{u}, 0, c_0)$ | $\left\langle W, Q, D_3, L_3, P_3, P_1 \mp \frac{E_0 d}{c_0} D_1, P_2 \pm \frac{E_0 d}{c_0} D_2 \right\rangle$ | $\langle W, Q, D_3, L_3 \rangle$ | $\left\langle P_1 \mp \frac{E_0 d}{c_0} D_1, P_2 \mp \frac{E_0 d}{c_0} D_2, P_3 \right\rangle$ |
| $6'''_0$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, \mathbf{0}, h\boldsymbol{u}, 0, c_0)$ | $\langle W, Q, D_3, L_3, P_3, P_1, P_2 \rangle$ | $\langle W, Q, D_3, L_3 \rangle$ | $\langle P_a \rangle$ |
| $8$ | $\mathbf{M}(q_0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \ell \boldsymbol{u}, E_0, 0)$ | $\langle W, H, Q, Z, L_3 \rangle$ | $\langle W, Q, Z, L_3 \rangle$ | $\langle H \rangle$ |
| $8'$ | $\mathbf{M}(0, 0, d\boldsymbol{u}, p\boldsymbol{u}^\perp, \mathbf{0}, E_0, 0)$ | $\langle W, H, Q, D_3, P_1 \rangle$ | $\langle W, Q, D_3 \rangle$ | $\langle H, P_1 \rangle$ |
| $8''$ | $\mathbf{M}(0, 0, d\boldsymbol{e}_3, p\sin\theta\boldsymbol{e}_2 - \frac{\epsilon_0 d}{c_0}\boldsymbol{e}_3, \mathbf{0}, 0, c_0)$ | $\left\langle W, Q, D_3, P_1 - \frac{\epsilon_0}{c_0} D_1, P_3 - \frac{p}{d}\sin\theta D_1 \right\rangle$ | $\langle W, Q, D_3 \rangle$ | $\left\langle P_1 - \frac{\epsilon_0}{c_0} D_1, P_3 - \frac{p}{d}\sin\theta D_1 \right\rangle$ |
| $8'''$ | $\mathbf{M}(c_0, 0, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0, c_0)$ | $\langle W, Q, L_a \rangle$ | $\langle W, Q, L_a \rangle$ | $0$ |
| $10$ | $\mathbf{M}(c_0, 0, \mathbf{0}, \mathbf{0}, \ell \boldsymbol{u}, 0, c_0)$ | $\langle W, Q, L_3 \rangle$ | $\langle W, Q, L_3 \rangle$ | $0$ |

**Table 2.** Stabilisers associated to particle dynamics

### 3.2 Geometric particle dynamics

We interpret the planon particle dynamics as taking place in an aristotelian spacetime whose momenta lie in the different coadjoint orbits of $G$. The discussion here follows that of [1, section 9] which itself follows the discussion in [21, appendix A.4] (see also e.g., [30–34] and references therein). As shown in that latter reference, the curves $\gamma(\tau)$ extremising the action functional in equation (3.1) are of the form $\gamma(\tau) = g_0 c(\tau)$, where $g_0 \in G$ and $c : I \to G_\alpha$ is any curve in the stabiliser of $\alpha$. Under the orbit map, $\pi_\alpha(\gamma(\tau)) = \operatorname{Ad}^*_{g_0} \alpha$, which is a fixed point in the coadjoint orbit $\mathcal{O}_\alpha$ of $\alpha$. The curve $\gamma$ can now be mapped to any other homogeneous space of $G$ via the corresponding orbit map. For example, let $M$ denote the aristotelian spacetime with (non-effective) Klein pair $(\mathfrak{g}, \mathfrak{g}_o)$, where $\mathfrak{g}_o = \langle L_a, D_a, Q, W, Z \rangle$. Let $G_o$ be the connected subgroup of $G$ with Lie algebra $\mathfrak{g}_o$ and let $o \in M$ be any point with stabiliser $G_o$. We may use the orbit map $\pi_o : G \to M$ to map the curves $\gamma(\tau)$ on $G$ to curves on $M$: these are the classical trajectories whose momenta lie in $\mathcal{O}_\alpha$.

If the curve $c$ landed in the intersection $G_\alpha \cap G_o$ then clearly this would correspond to a trajectory which is fixed at the point $g_0 \cdot o \in M$. Hence we can choose a complement $\mathfrak{m}$ to $\mathfrak{g}_\alpha \cap \mathfrak{g}_o$ in $\mathfrak{g}_\alpha$, so that $\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{g}_o) \oplus \mathfrak{m}$ and give local coordinates to $G_\alpha$ by choosing bases for $\mathfrak{g}_\alpha \cap \mathfrak{g}_o$ and for $\mathfrak{m}$ and write elements in $G_\alpha$ locally as a product of exponentials $\exp(X)\exp(Y)$ with $Y \in \mathfrak{g}_\alpha \cap \mathfrak{g}_o$ and $X \in \mathfrak{m}$. The local coordinates are then the coefficients of $X$ and $Y$ relative to the chosen bases. The complement $\mathfrak{m}$ detects the possible directions along which the particle trajectory can evolve.

In table 2 we list the stabiliser algebras $\mathfrak{g}_\alpha$ for each type of coadjoint orbit in table 1, the intersection $\mathfrak{g}_\alpha \cap \mathfrak{g}_o$ and a choice of vector space complement $\mathfrak{m}$ such that $\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{g}_o) \oplus \mathfrak{m}$.

One can read off from this table, that particles corresponding to orbits with $c = c_0 \neq 0$ do not evolve in time: as the time translation generator $H$ is not in the stabiliser of $\alpha$: this just reflects that energy is not conserved for such orbits and is indeed unbounded from both below and above. The remaining orbits, labelled $0, 2, 4, 6, 6', 8$ and $8'$, are studied in turn below.



JHEP11(2025)030

We can read off some properties of those orbits from the table: particles with momenta in coadjoint orbits of types $0, 2$ are unrestricted; those in orbits of types $4, 8'$ move along "lines"; those in orbits of type $6'$ move along "planes"; whereas those in orbits of types $6, 8$ do not move. These statements do not necessarily mean that mobility is restricted, but they simply describe the existence of such particle trajectories. This is analogous to how a massive (galilean or minkowskian) particle can be boosted to its rest frame, where it appears as if it does not move.

### 3.3 Particle dynamics

Based on our understanding of the coadjoint orbits, we will now derive actions for the orbits where $c = 0$, and we shall analyse their mobility restrictions. Since with the standard planon interpretation of the generators a nonzero $c$ results in an energy that is unbounded from below, we will exclude such cases from the current analysis.

Since $c = 0$ the coadjoint orbits and actions can again be mapped to the Bargmann case (see, e.g., [1]), although with different physical interpretation. In particular, some of the orbits most relevant in the context of planons, such as those describing dipoles, correspond to exotic massless orbits in the Bargmann framework.

In the following, to simplify the interpretation of the orbits, we will replace $\boldsymbol{a}$ with $\boldsymbol{x}$, where $\boldsymbol{x}$ represents the position of the corresponding particle and the orbit numbers and representatives are taken from table 1.

#### 3.3.1 Orbit #0 (trace particle)

Let us consider a coadjoint orbit with a representative given by

$$\alpha = \mathbf{M}(0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \mathbf{0}, E_0, 0)\,, \tag{3.7}$$

where $E_0$ and $\tilde{q}_0$ denote the energy and the trace charge, respectively.

The Lagrangian is explicitly obtained from the Maurer-Cartan form (3.6), resulting in

$$L\,[\tilde{\varphi}, t] = \tilde{q}_0 \dot{\tilde{\varphi}} - E_0 \dot{t}\,. \tag{3.8}$$

Since this Lagrangian is a boundary term, there is no dynamic associated with it. Given that the stabiliser is the entire group, this "trace particle" can also be interpreted as a vacuum configuration. Indeed, from the perspective of the Bargmann group, this orbit is viewed as a vacuum (see for example, [1]). It is important to note that the symmetries impose no restrictions on the "mobility."

#### 3.3.2 Orbit #2 (spinning trace particle)

Let us consider the following representative:

$$\alpha = \mathbf{M}(0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \boldsymbol{j}, E_0, 0) \tag{3.9}$$

where $\|\boldsymbol{j}\| = \ell$. The Lagrangian associated with this coadjoint orbit is given by

$$L\,[\tilde{\varphi}, t, R\,(\boldsymbol{\phi})] = \tilde{q}_0 \dot{\tilde{\varphi}} - E_0 \dot{t} + \boldsymbol{j} \cdot \varepsilon^{-1}(R^{-1}dR)\,. \tag{3.10}$$

Note that only the spin part of the Lagrangian is not a boundary term. Consequently, this orbit possesses only spin degrees of freedom.

Following [21], it is useful to adopt the following parametrisation for the rotation matrix:

$$R(\boldsymbol{\phi}) = e^{\phi_1 \varepsilon_1} e^{\phi_2 \varepsilon_2} e^{\phi_3 \varepsilon_3}, \tag{3.11}$$

with $(\varepsilon_a)_{bc} = -\varepsilon_{abc}$. In the particular case where the angular momentum is aligned with the $z$-axis, we can write:

$$L[\tilde{\varphi}, t, \boldsymbol{\phi}] = \tilde{q}_0 \dot{\tilde{\varphi}} - E_0 \dot{t} + \ell \left( \dot{\phi}_3 + \sin(\phi_2) \dot{\phi}_1 \right) . \tag{3.12}$$

If one removes the boundary terms, the Lagrangian can be rewritten as

$$L\left[\Pi^1, \phi_1\right] = \Pi^1 \dot{\phi}_1, \tag{3.13}$$

where $\Pi^1 := \ell \sin\phi_2$ is the canonical conjugate to $\phi_1$.

In the context of the Bargmann group interpretation, this configuration represents a "spinning vacuum."

### 3.3.3 Orbit #4 (spinning trace particle with momentum)

For this orbit, it is necessary to consider the following representative:

$$\alpha = \mathbf{M}(0, \tilde{q}_0, \mathbf{0}, \boldsymbol{p}, \boldsymbol{j}, E_0, 0), \tag{3.14}$$

where $\|\boldsymbol{p}\|^2 = p_0^2$ is a constant, with $p_0 > 0$. Additionally, $\boldsymbol{j} \cdot \boldsymbol{p} = p_0 h_0$ for a real constant $h_0$. As a consequence, we can express

$$\boldsymbol{p} = p_0 \hat{\boldsymbol{n}}, \tag{3.15}$$

where

$$\hat{\boldsymbol{n}} = (\sin\phi_2, -\sin\phi_1 \cos\phi_2, -\cos\phi_1 \cos\phi_2) . \tag{3.16}$$

The Lagrangian corresponding to this orbit is then given by

$$L[\tilde{\varphi}, t, \boldsymbol{x}, \boldsymbol{\phi}] = \tilde{q}_0 \dot{\tilde{\varphi}} + \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - E_0 \dot{t} + \boldsymbol{j} \cdot \varepsilon^{-1}(R^{-1} dR). \tag{3.17}$$

Assuming that $\boldsymbol{j}$ and $\boldsymbol{p}$ are both aligned along the $z$-axis, such that $\boldsymbol{j} = (0, 0, j_0)$ and $\boldsymbol{p} = (0, 0, p_0)$, the Lagrangian can be expressed as

$$L[\tilde{\varphi}, t, \boldsymbol{x}, \boldsymbol{\phi}] = \tilde{q}_0 \dot{\tilde{\varphi}} + h_0 \left( \dot{\phi}_3 + \sin(\phi_2) \dot{\phi}_1 \right) + p_0 \hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{x}} - E_0 \dot{t}. \tag{3.18}$$

If the boundary terms are discarded, and the following canonical momenta are introduced:

$$\boldsymbol{\pi} := \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = p_0 \hat{\boldsymbol{n}} \qquad\qquad \Pi^1 := \frac{\partial L}{\partial \dot{\phi}_1} = h_0 \sin\phi_2, \tag{3.19}$$

which satisfy the constraints

$$\|\boldsymbol{\pi}\|^2 - p_0^2 = 0 \qquad\qquad p_0 \Pi^1 - h_0 \pi_1 = 0, \tag{3.20}$$

then the Lagrangian in canonical form can be written as

$$L_{\text{can}}\left[\boldsymbol{x},\boldsymbol{\pi},\phi_1,\Pi^1,\eta,\eta_1\right]=\Pi^1\dot{\phi}_1+\boldsymbol{\pi}\cdot\dot{\boldsymbol{x}}-\eta\left(\|\boldsymbol{\pi}\|^2-p_0^2\right)-\eta_1\left(p_0\Pi^1-h_0\pi_1\right). \tag{3.21}$$

The dynamics of this particle can be understood in terms of carrollian, and galilean physics. In the carrollian context, the dynamics arising from this Lagrangian were explored in section 3.4 of [21], where it was referred to as "massless carrollion." This Lagrangian corresponds to one of the massless coadjoint orbits of the Carroll group. The specific case where the spin vanishes had been previously studied in [35]. On the other hand, this Lagrangian appears in one of the massless representations of the Bargmann group, where the position of the particle is replaced by its velocity. This case was analysed in section 8.3.3 of [1].

#### 3.3.4 Orbit #6 (monopole)

The coadjoint orbit in this case is determined by the following representative:

$$\alpha=\mathbf{M}\left(q_0,\tilde{q}_0,\mathbf{0},\mathbf{0},\mathbf{0},E_0,0\right). \tag{3.22}$$

The Lagrangian is given by

$$L\left[\varphi,\tilde{\varphi},\boldsymbol{\beta},\boldsymbol{x},t\right]=-E_0\dot{t}+q_0\dot{\varphi}+\tilde{q}_0\dot{\tilde{\varphi}}-\frac{1}{2}q_0\|\boldsymbol{x}\|^2\dot{\tilde{\varphi}}-q_0\boldsymbol{x}\cdot\dot{\boldsymbol{\beta}}. \tag{3.23}$$

We fix the gauge $\tau=t$ and ignore the remaining boundary terms. Furthermore, it is useful to perform integration by parts on the last two terms. The Lagrangian can then be expressed as follows, where the derivatives are now with respect to the physical time $t$

$$L\left[\tilde{\varphi},\boldsymbol{\beta},\boldsymbol{x}\right]=q_0\left(\boldsymbol{x}\tilde{\varphi}+\boldsymbol{\beta}\right)\cdot\dot{\boldsymbol{x}}. \tag{3.24}$$

The Lagrangian can be written in Hamiltonian form by introducing the conjugate variable to $\boldsymbol{x}$, defined as

$$\boldsymbol{\pi}:=\frac{\partial L}{\partial\dot{\boldsymbol{x}}}=q_0\left(\boldsymbol{x}\tilde{\varphi}+\boldsymbol{\beta}\right). \tag{3.25}$$

The Lagrangian then becomes

$$L\left[\boldsymbol{x},\boldsymbol{\pi}\right]=\boldsymbol{\pi}\cdot\dot{\boldsymbol{x}}. \tag{3.26}$$

The equations of motion are given by

$$\dot{\boldsymbol{x}}=\mathbf{0}\qquad\qquad\dot{\boldsymbol{\pi}}=\mathbf{0} \tag{3.27}$$

showing that this particle cannot move. Although we refer to it as a monopole, it actually carries both electric and quadrupole charges.

It is interesting to contrast the monopole with the corresponding Bargmann case, which is the well-known massive Galilei particle (see appendix A for more details concerning the dictionary). The starting point is the action

$$L[a_+,t,\boldsymbol{x},\boldsymbol{v}]=0+m\dot{a}_+-E_0\dot{t}-\frac{1}{2}m\|\boldsymbol{v}\|^2\dot{t}+m\boldsymbol{v}\cdot\dot{\boldsymbol{x}}, \tag{3.28}$$

which agrees with (3.23), up to the first term which has no counterpart in the Galilei case, and immaterial changes of signs. Again we gauge fix the physical time $\tau = t$ and ignore the remaining boundary terms, which leads to

$$L[\boldsymbol{x}, \boldsymbol{v}] = -\frac{1}{2} m \|\boldsymbol{v}\|^2 + m \boldsymbol{v} \cdot \dot{\boldsymbol{x}} \,, \tag{3.29}$$

that must be contrasted with (3.24). In particular, varying (3.29) with respect to $\boldsymbol{x}$ leads to

$$\dot{\boldsymbol{v}} = \boldsymbol{0} \,, \tag{3.30}$$

which is the galilean analog of $\dot{\boldsymbol{x}} = \boldsymbol{0}$ of (3.27).

We can now vary (3.29) with respect to $\boldsymbol{v}$, which leads to the equation $\boldsymbol{v} = \dot{\boldsymbol{x}}$. We are then allowed to substitute this back into the action to obtain

$$L[\boldsymbol{x}] = \frac{1}{2} m \|\dot{\boldsymbol{x}}\|^2 \,. \tag{3.31}$$

As is well known, these particles move along straight lines.

This example shows that the physical interpretation of the parameters do play an important role for the mobility the particle models.

#### 3.3.5 Orbit #8 (spinning monopole)

Let us consider the following representative of the coadjoint orbit

$$\alpha = \mathsf{M} \left( q_0, \tilde{q}_0, 0, 0, \boldsymbol{j}, E_0, 0 \right) , \tag{3.32}$$

where $\|\boldsymbol{j}\|^2 = \ell^2$ is a constant.

The Lagrangian is given by

$$L \left[ \varphi, \tilde{\varphi}, \boldsymbol{x}, \boldsymbol{\beta}, t, R\left(\boldsymbol{\phi}\right) \right] = q_0 \dot{\varphi} + \left( \tilde{q}_0 - \frac{1}{2} q_0 \|\boldsymbol{x}\|^2 \right) \dot{\tilde{\varphi}} - q_0 \boldsymbol{x} \cdot \dot{\boldsymbol{\beta}} - E_0 \dot{t} + \boldsymbol{j} \cdot \varepsilon^{-1} (R^{-1} dR). \tag{3.33}$$

Using the parametrisation for the rotation matrix in (3.11), the spin part becomes

$$L_{\text{spin}} \left( \boldsymbol{\phi} \right) = \ell \left( \dot{\phi}_3 + \sin \phi_2 \right) \dot{\phi}_1. \tag{3.34}$$

By introducing the canonical momenta as defined in eq. (3.19) and neglecting boundary terms, the Lagrangian in canonical form can be expressed as:

$$L \left[ \boldsymbol{x}, \boldsymbol{\pi}, \phi_1, \Pi^1 \right] = \Pi^1 \dot{\phi}_1 + \boldsymbol{\pi} \cdot \dot{\boldsymbol{x}}. \tag{3.35}$$

This action depends on 8 independent canonical variables, which coincides with the dimension of the coadjoint orbit. In the context of the Bargmann interpretation, it describes a massive spinning particle.

### 3.3.6 Orbit #6' (dipole)

Let us consider the following representative:

$$\alpha = \mathbf{M}\,(0, 0, \boldsymbol{d}, \mathbf{0}, \mathbf{0}, E_0, 0)\,, \tag{3.36}$$

where $\|\boldsymbol{d}\|^2 = d_0^2$ with $d_0$ being a constant.

The Lagrangian is given by

$$L\,[\boldsymbol{\phi}, \boldsymbol{x}, \boldsymbol{\beta}, \tilde{\varphi}, t] = (R\boldsymbol{d}) \cdot \boldsymbol{x}\dot{\tilde{\varphi}} + (R\boldsymbol{d}) \cdot \dot{\boldsymbol{\beta}} - E_0 \dot{t}\,. \tag{3.37}$$

It is useful to define the following fields:

$$\boldsymbol{D} := R\boldsymbol{d}_0 \qquad\qquad \tilde{Q} := \boldsymbol{D} \cdot \boldsymbol{x}\,, \tag{3.38}$$

where the field $\boldsymbol{D}$ represents the dipole moment obtained by applying a generic rotation to the representative dipole moment $\boldsymbol{d}$. The field $\tilde{Q}$ can be interpreted as describing the degree of freedom associated with the trace charge. They satisfy the following constraints

$$\|\boldsymbol{D}\|^2 - d_0^2 \approx 0 \qquad\qquad \tilde{Q} - \boldsymbol{D} \cdot \boldsymbol{x} \approx 0\,. \tag{3.39}$$

Thus, we can write

$$\begin{aligned} L\,[t, E, \tilde{\varphi}, \boldsymbol{D}, \boldsymbol{\beta}, \boldsymbol{x}, \boldsymbol{\pi}, \eta_1, \eta_2, \boldsymbol{\eta}_3, N] = &- E\dot{t} + \tilde{Q}\dot{\tilde{\varphi}} + \boldsymbol{D} \cdot \dot{\boldsymbol{\beta}} + \boldsymbol{\pi} \cdot \dot{\boldsymbol{x}} \\ &- \eta_1 \mathcal{C}_1 - \eta_2 \mathcal{C}_2 - \boldsymbol{\eta}_3 \cdot \boldsymbol{\mathcal{C}}_3 - N\mathcal{C}_4\,, \end{aligned} \tag{3.40}$$

with the constraints

$$\mathcal{C}_1 := \|\boldsymbol{D}\|^2 - d_0^2 \approx 0 \tag{3.41}$$

$$\mathcal{C}_2 := \tilde{Q} - \boldsymbol{D} \cdot \boldsymbol{x} \approx 0 \tag{3.42}$$

$$\boldsymbol{\mathcal{C}}_3 := \boldsymbol{\pi} \approx \mathbf{0} \tag{3.43}$$

$$\mathcal{C}_4 := E - E_0 \approx 0\,. \tag{3.44}$$

The equations of motion are given by

$$\dot{\tilde{\varphi}} = \eta_2 \qquad\qquad \dot{\tilde{Q}} = 0 \tag{3.45}$$

$$\dot{\boldsymbol{\beta}} = 2\eta_1 \boldsymbol{D} - \eta_2 \boldsymbol{x} \qquad\qquad \dot{\boldsymbol{D}} = 0 \tag{3.46}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{\eta}_3 \qquad\qquad \dot{\boldsymbol{\pi}} = \eta_2 \boldsymbol{D} \tag{3.47}$$

$$\dot{t} = -N \qquad\qquad \dot{E} = 0\,. \tag{3.48}$$

The time derivative of the constraints do not generate secondary constraints. However, it imposes conditions on certain Lagrange multipliers, indicating that some of the constraints are of second-class. Indeed, the time derivative of $\boldsymbol{\mathcal{C}}_3$ yields $\eta_2 = 0$, while the time derivative of $\mathcal{C}_2$ gives

$$\boldsymbol{D} \cdot \dot{\boldsymbol{x}} = \boldsymbol{D} \cdot \boldsymbol{\eta}_3 = 0\,. \tag{3.49}$$

Thus, the velocity component in the direction of the dipole moment vanishes, a key characteristic of a planon. Moreover, as we will show below, the components of $\boldsymbol{x}$ perpendicular to the dipole moment transform under gauge transformations, indicating that the dipole can move freely within the plane orthogonal to its dipole moment.

To complete the analysis, we will classify the constraints into first and second-class constraints. The second-class constraints are represented by the pair:

$$\chi_1 := \mathcal{C}_1 = \tilde{Q} - \boldsymbol{D} \cdot \boldsymbol{x} \qquad\qquad \chi_2 := \boldsymbol{D} \cdot \boldsymbol{\pi} \,. \tag{3.50}$$

Indeed, they obey the following Poisson bracket:

$$\{\chi_1, \chi_2\} = -d_0^2 \neq 0. \tag{3.51}$$

On the other hand, the first-class constraints are given by

$$\mathcal{C}_1 = \|\boldsymbol{D}\|^2 - d_0^2 \approx 0 \qquad\qquad \boldsymbol{\pi}^T := \boldsymbol{\pi} - \frac{1}{d_0^2} (\boldsymbol{D} \cdot \boldsymbol{\pi}) \boldsymbol{D} \approx \boldsymbol{0} \,. \tag{3.52}$$

Notice that the constraint $\boldsymbol{\mathcal{C}}_3 = \boldsymbol{\pi} \approx 0$ was split between its second-class part $\chi_2$ and its first-class part $\boldsymbol{\pi}^T$. As mentioned above, the first-class constraints generate, via $G = \boldsymbol{\lambda}(t) \cdot \boldsymbol{\pi}^T$, gauge transformations of the coordinates $\delta_\lambda \boldsymbol{x} = \{\boldsymbol{x}, G\} = \boldsymbol{\lambda}^T$, which are perpendicular to the dipole moment.

Given the 16 canonical variables, 4 first-class constraints, and 2 second-class constraints, the number of independent degrees of freedom is $16 - 2 \times 4 - 2 = 6$, which matches the dimension of the orbit.

The Lagrangian simplifies when the second-class constraints are enforced to vanish strongly, $\chi_1 = \chi_2 = 0$. By further applying the gauge-fixing conditions $\tau = t$ and $\boldsymbol{x}^T = \boldsymbol{0}$, and solving some of the first-class constraints, we obtain:

$$L\left[\boldsymbol{x}, \tilde{\varphi}, \boldsymbol{D}, \boldsymbol{\beta}\right] = d_0 x^L \dot{\tilde{\varphi}} + \boldsymbol{D} \cdot \dot{\boldsymbol{\beta}} - \eta_1 \left(\|\boldsymbol{D}\|^2 - d_0^2\right), \tag{3.53}$$

where $x^L := \frac{\boldsymbol{D}}{d_0} \cdot \boldsymbol{x}$.

In the context of the Bargmann group interpretation, this orbit is discussed in section 8.3.4 of [1] and corresponds to a massless orbit describing a galilean particle that does not evolve in time. The orbit is defined instantaneously at a fixed value of the physical time $t$. In contrast, the planon case has a non-trivial time evolution.

### 3.3.7 Orbit #8' (dipole with momentum)

Let us consider the following representative:

$$\alpha = \mathbf{M}\left(0, 0, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{0}, E_0, 0\right), \tag{3.54}$$

where $\|\boldsymbol{d}\|^2 = d_0^2$ and $\boldsymbol{d} \cdot \boldsymbol{p} = 0$, with $\|\boldsymbol{d} \times \boldsymbol{p}\| = d_0 p_0 > 0$.

The Lagrangian is given by

$$L\left[R\left(\boldsymbol{\phi}\right), \boldsymbol{x}, \boldsymbol{\beta}, \tilde{\varphi}, t\right] = (R\boldsymbol{d}) \cdot \boldsymbol{x} \dot{\tilde{\varphi}} + (R\boldsymbol{d}) \cdot \dot{\boldsymbol{\beta}} + R\boldsymbol{p} \cdot \dot{\boldsymbol{x}} - E_0 \dot{t} \,. \tag{3.55}$$

It is convenient to introduce the following variables:

$$\boldsymbol{D} := R\boldsymbol{d} \qquad \boldsymbol{\pi} := R\boldsymbol{p} \qquad \tilde{Q} := \boldsymbol{D}\cdot\boldsymbol{x}, \tag{3.56}$$

The Lagrangian in Hamiltonian form can then be written as

$$\begin{aligned} L\left[E,t,\tilde{Q},\tilde{\varphi},\boldsymbol{D},\boldsymbol{\beta},\boldsymbol{\pi},\boldsymbol{x},N,\eta_1,\eta_2,\eta_3,\eta_4\right] = &- E\dot{t} + \tilde{Q}\dot{\tilde{\varphi}} + \boldsymbol{D}\cdot\dot{\boldsymbol{\beta}} + \boldsymbol{\pi}\cdot\dot{\boldsymbol{x}} - \eta_1\mathcal{C}_1 - \eta_2\mathcal{C}_2 \\ &- \eta_3\mathcal{C}_3 - \eta_4\mathcal{C}_4 - N\mathcal{C}_5\,, \end{aligned} \tag{3.57}$$

with the following constraints

$$\mathcal{C}_1 := \tilde{Q} - \boldsymbol{D}\cdot\boldsymbol{x} \approx 0 \tag{3.58}$$
$$\mathcal{C}_2 := \|\boldsymbol{D}\|^2 - d_0^2 \approx 0 \tag{3.59}$$
$$\mathcal{C}_3 := \|\boldsymbol{\pi}\|^2 - p_0^2 \approx 0 \tag{3.60}$$
$$\mathcal{C}_4 := \boldsymbol{D}\cdot\boldsymbol{\pi} \approx 0 \tag{3.61}$$
$$\mathcal{C}_5 := E - E_0 \approx 0\,. \tag{3.62}$$

Note that the constraints $\mathcal{C}_1$ and $\mathcal{C}_4$ are of second-class. Indeed, the obey the following Poisson bracket:

$$\{\mathcal{C}_1,\mathcal{C}_4\} = -d_0^2 \neq 0\,. \tag{3.63}$$

The preservation in time of these second-class constraints implies that the corresponding Lagrange multipliers must satisfy

$$\eta_1 = \eta_4 = 0\,. \tag{3.64}$$

Consequently, the equations of motion become

$$\dot{t} = -N \qquad \dot{E} = 0 \tag{3.65}$$
$$\dot{\tilde{\varphi}} = 0 \qquad \dot{\tilde{Q}} = 0 \tag{3.66}$$
$$\dot{\boldsymbol{\beta}} = 2\eta_2\boldsymbol{D} \qquad \dot{\boldsymbol{D}} = 0 \tag{3.67}$$
$$\dot{\boldsymbol{x}} = 2\eta_3\boldsymbol{\pi} \qquad \dot{\boldsymbol{\pi}} = 0\,. \tag{3.68}$$

An immediate consequence to note is that

$$\boldsymbol{D}\cdot\dot{\boldsymbol{x}} = 0\,. \tag{3.69}$$

Therefore, the elementary dipole cannot move in the direction of its dipole moment. It behaves as a planon.

The second-class constraints $\mathcal{C}_1 = 0$ and $\mathcal{C}_4 = 0$ can be solved by decomposing $\boldsymbol{\pi}$ into its longitudinal and transverse components relative to the dipole moment. Consequently, one can write

$$\tilde{Q} = d_0 x^L \qquad \pi^L = 0\,, \tag{3.70}$$

where $x^L := \frac{\boldsymbol{D}}{d_0} \cdot \boldsymbol{x}$ and $\pi^L := \frac{\boldsymbol{D}}{d_0} \cdot \boldsymbol{\pi}$. If, in addition, the gauge condition $\tau = t$ is imposed, the Lagrangian can be expressed as

$$L\left[\tilde{\varphi}, \boldsymbol{D}, \boldsymbol{\beta}, \boldsymbol{\pi}, \boldsymbol{x}, \eta_2, \eta_3\right] = d_0 x^L \dot{\tilde{\varphi}} + \boldsymbol{D} \cdot \dot{\boldsymbol{\beta}} + \boldsymbol{\pi}^T \cdot \dot{\boldsymbol{x}} - \eta_2 \left(\|\boldsymbol{D}\|^2 - d_0^2\right) - \eta_3 \left(\|\boldsymbol{\pi}\|^2 - p_0^2\right) .$$

This orbit was analysed in the context of the Bargmann group in section 8.3.5 of [1] and corresponds to a massless representation without time evolution.

Let us study in detail the motion in the plane orthogonal to the dipole moment. Denoting its coordinates by $x$ and $y$, and assuming that $\pi_x, \pi_y \neq 0$, from eq. (3.68) it follows that

$$\frac{\dot{x}}{\pi_x} = \frac{\dot{y}}{\pi_y}. \tag{3.71}$$

Therefore, integrating over time, one obtains the following trajectory:

$$y = \frac{\pi_y}{\pi_x} x + \bar{y}, \tag{3.72}$$

where $\bar{y}$ is a constant. Note that the above equation is gauge invariant, as it does not depend on $\eta_3$, and it shows that the particle moves in a straight line within the plane orthogonal to the dipole moment, which is consistent with the result obtained in section 3.2. On the other hand, when either $\pi_x$ or $\pi_y$ vanishes, the trajectory is also a straight line, aligned with the direction of the non-vanishing momentum.

## 4 Composite dipoles

In this section, we construct a Lagrangian that describes the dynamics of a non-elementary dipoles[4] that naturally couple to fracton traceless gauge field [23], analyse their symmetries and first quantise them.

First, we derive the Lagrangian for the dipole based solely on the symmetries of the problem. The action has planon symmetries, but also transversal galilean and longitudinal carrollian symmetries. We also show how to derive the dynamics of these non-elementary dipoles by assembling elementary monopoles (excitations of orbit #6) by using an appropriate interaction potential. These dipoles can be coupled to a fracton traceless gauge field [23] and we thereby recover the generalised force law of Pretko from a variational symmetry.

We also first quantise the resulting system resulting in a gaussian Schrödinger-like field theory that was already discussed in the context of the fracton/elasticity duality [24, 25]. We also show that the free theory has again the mixed Carroll-Galilei symmetries.

### 4.1 Dynamics of non-elementary dipoles from symmetries

The charge density of a dipole with position $\boldsymbol{z}(t)$ and dipole moment $\boldsymbol{d}$ takes the form

$$\rho(t, \boldsymbol{x}) = -d^i \frac{\partial}{\partial x^i} \delta\left(\boldsymbol{x} - \boldsymbol{z}(t)\right) \qquad J_{ij}(t, \boldsymbol{x}) = -d_{(i} \dot{z}_{j)} \delta\left(\boldsymbol{x} - \boldsymbol{z}(t)\right). \tag{4.1}$$

[4] To avoid confusion let us note that the composite dipoles we discuss are not elementary systems of the planon group. They are ideal or point electric dipoles which can be thought of as limits of the physical dipoles where the distance goes to zero and the charge to infinity such that the dipole moment stays finite (see, e.g., [36]).

As discussed in section 1 the generators $\boldsymbol{D} = \boldsymbol{d}$ and $Z = \boldsymbol{d} \cdot \boldsymbol{z}$ depend exclusively on the canonical variables $\boldsymbol{d}$ and $\boldsymbol{z}$. As a result, the symmetries only act on their corresponding canonical conjugates $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$. Using the following canonical Poisson brackets

$$\{z^i, \pi_j\} = \delta^i_j \qquad \{d^i, \sigma_j\} = \delta^i_j \,, \tag{4.2}$$

it can be shown that the transformation laws under quadrupole transformations with parameter $\epsilon$ and dipole transformations with parameter $\boldsymbol{\alpha}$ are given by

$$\delta \boldsymbol{z} = \boldsymbol{0} \qquad \delta \boldsymbol{\pi} = -\epsilon \boldsymbol{d} \tag{4.3a}$$

$$\delta \boldsymbol{d} = \boldsymbol{0} \qquad \delta \boldsymbol{\sigma} = -\epsilon \boldsymbol{z} - \boldsymbol{\alpha} \,. \tag{4.3b}$$

It is useful to decompose the momentum $\boldsymbol{\pi} = \boldsymbol{\pi}^T + \boldsymbol{\pi}^L$ into its transverse and longitudinal components relative to the dipole moment $\boldsymbol{d}$

$$\pi^T_i = P_{ij} \pi^j \qquad \pi^L_i = P^L_{ij} \pi^j \,, \tag{4.4}$$

where we used the transverse and longitudinal projectors

$$P_{ij} = \delta_{ij} - \hat{d}_i \hat{d}_j \qquad P^L_{ij} = \hat{d}_i \hat{d}_j \,, \tag{4.5}$$

with $\hat{d}_i$ being the unit vector in the direction of the dipole moment $\hat{\boldsymbol{d}} = \boldsymbol{d}/d$, and $d$ is its magnitude. They are orthogonal $P_{ij} P^L_{jk} = 0$. In particular, it is evident that only the longitudinal component of the momentum transforms under (4.3a)

$$\delta \boldsymbol{\pi}^T = \boldsymbol{0} \qquad \delta \boldsymbol{\pi}^L = -\epsilon \boldsymbol{d} \,. \tag{4.6}$$

Thus, to construct a Lagrangian for a free dipole that remains invariant under planon symmetry, one may choose a Hamiltonian proportional to $(\boldsymbol{\pi}^T)^2 = \boldsymbol{\pi}^2 - (\hat{\boldsymbol{d}} \cdot \boldsymbol{\pi})^2$. This choice yields the following Lagrangian (in Hamiltonian form)

$$L_{\text{dip}} [\boldsymbol{z}, \boldsymbol{\pi}, \boldsymbol{\sigma}, \boldsymbol{d}] = \pi_i \dot{z}^i + \sigma_i \dot{d}^i - \frac{1}{2m} P_{ij} \pi^i \pi^j \,. \tag{4.7}$$

The corresponding equations of motion are then given by

$$\dot{z}^i = \frac{1}{m} P^{ij} \pi_j \tag{4.8a}$$

$$\dot{\pi}_i = 0 \tag{4.8b}$$

$$\dot{d}^i = 0 \tag{4.8c}$$

$$\dot{\sigma}_i = \frac{\hat{\boldsymbol{d}} \cdot \boldsymbol{\pi}}{md} P_{ij} \pi^j \,. \tag{4.8d}$$

Note that contracting equation (4.8a) with $d_i$ yields

$$\boldsymbol{d} \cdot \dot{\boldsymbol{z}} = 0 \,, \tag{4.9}$$

which is precisely the condition that ensures the conservation of the trace of the quadrupole moment, cf. (1.4), and defines the property of being a planon. Let us also emphasise that the conservation of the dipole moment, $\dot{\boldsymbol{d}} = \boldsymbol{0}$, is a consequence of the action principle and not put in by hand. The (im)mobility is particularly transparent when we also decompose the position of the dipole into transversal and longitudinal components $\boldsymbol{z} = \boldsymbol{z}^T + \boldsymbol{z}^L$ and use equations (4.8a) and (4.8b) to obtain

$$\ddot{\boldsymbol{z}}^T = \boldsymbol{0} \qquad\qquad \dot{\boldsymbol{z}}^L = \boldsymbol{0}\,. \tag{4.10}$$

We see that the dipole can move freely transversal to the dipole moment, but the longitudinal component is fixed (see figure 1).

### 4.2 Composite dipole from the interaction of two elementary monopoles

In this section, we shall derive the Lagrangian for the composite dipole (4.7) by considering the interaction of two elementary monopoles described by the orbit #6. The interaction term is selected to ensure the invariance of the Lagrangian under the diagonal subgroup of the direct sum of the planon algebras associated with each elementary particle.

Let us consider the Lagrangian for two elementary particles of the orbit #6 with opposite electric and quadrupole charges. According to (3.26), it can be written as

$$L_{\text{dip}} = \boldsymbol{\pi}_1 \cdot \dot{\boldsymbol{x}}_1 + \boldsymbol{\pi}_2 \cdot \dot{\boldsymbol{x}}_2 - V\,. \tag{4.11}$$

Here, $V$ represents the potential that enables the dipole to exist as a bound state of two elementary particles.

According to (3.25), the canonical momenta are defined as

$$\boldsymbol{\pi}_1 = q_0\left(\boldsymbol{x}_1\tilde{\phi}_1 + \boldsymbol{b}_1\right) \qquad\qquad \boldsymbol{\pi}_2 = -q_0\left(\boldsymbol{x}_2\tilde{\phi}_2 + \boldsymbol{b}_2\right)\,. \tag{4.12}$$

In particular, under dipole transformations with parameters $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ and quadrupole transformations with parameters $\epsilon_1, \epsilon_2$, the canonical variables transform according to

$$\delta\boldsymbol{x}_1 = 0 \qquad\qquad \delta\boldsymbol{\pi}_1 = q_0\left(\boldsymbol{\alpha}_1 + \epsilon_1\boldsymbol{x}_1\right) \tag{4.13a}$$

$$\delta\boldsymbol{x}_2 = 0 \qquad\qquad \delta\boldsymbol{\pi}_2 = -q_0\left(\boldsymbol{\alpha}_2 + \epsilon_2\boldsymbol{x}_2\right)\,. \tag{4.13b}$$

It is convenient to re-express the fields in the Lagrangian in terms of the following quantities:

$$\boldsymbol{z} := \frac{1}{2}\left(\boldsymbol{x}_1 + \boldsymbol{x}_2\right) \qquad\qquad \boldsymbol{\pi} := \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 \tag{4.14a}$$

$$\boldsymbol{d} := q_0\left(\boldsymbol{x}_1 - \boldsymbol{x}_2\right) \qquad\qquad \boldsymbol{\sigma} := \frac{1}{2q_0}\left(\boldsymbol{\pi}_1 - \boldsymbol{\pi}_2\right)\,. \tag{4.14b}$$

Here, $\boldsymbol{z}$ denotes the average position of the two elementary particles, which describes the position of the dipole; $\boldsymbol{\pi}$ represents the total momentum; and $\boldsymbol{d}$ is the dipole moment of the system. The variable $\boldsymbol{\sigma}$, conjugate to $\boldsymbol{d}$, corresponds to the momentum difference between the two elementary particles. A pure dipole corresponds to the limit where the distance between the elementary particles approaches zero ($\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \to 0$) while their electric charge tends to infinity ($q_0 \to \infty$), such that the product remains constant in this limit.

The Lagrangian can then be rewritten as

$$L_{\text{dip}} = \boldsymbol{\pi} \cdot \dot{\boldsymbol{z}} + \boldsymbol{\sigma} \cdot \dot{\boldsymbol{d}} - V\,. \tag{4.15}$$



JHEP11(2025)030

Note that under the diagonal subgroup, with parameters $\epsilon = -\frac{1}{2}(\epsilon_1 + \epsilon_2)$ for the quadrupole transformations, and $\boldsymbol{\alpha} = -\frac{1}{2}(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)$ for the dipole transformations, the fields transform as follow

$$\delta \boldsymbol{z} = 0 \qquad \delta \boldsymbol{\pi} = -\epsilon \boldsymbol{d} \tag{4.16a}$$

$$\delta \boldsymbol{d} = 0 \qquad \delta \boldsymbol{\sigma} = -\epsilon \boldsymbol{z} - \boldsymbol{\alpha}\,. \tag{4.16b}$$

These are precisely the transformations laws for the composite dipole that were derived in (4.3).

The potential must be selected to ensure the invariance under the diagonal planon subgroup, in particular under the transformations (4.16).[5] Thus, it can be defined as

$$V = \frac{1}{2m} P_{ij} \pi^i \pi^j. \tag{4.17}$$

This term precisely corresponds to that appearing in the Lagrangian of the composite dipole in (4.7). In terms of the original variables of the elementary excitations, it is expressed as follows

$$V = \frac{1}{2m} \left[ \delta_{ij} - \frac{\left(x_1^i - x_2^i\right)\left(x_1^j - x_2^j\right)}{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2} \right] \left(\pi_1^i + \pi_2^i\right)\left(\pi_1^j + \pi_2^j\right). \tag{4.18}$$

It is important to emphasise that the presence of the potential is essential for enabling the dipole to move. In the absence of this potential, the system would only describe two immobile decoupled monopoles. The non-trivial dynamics, and in particular the planeon property of the composite system, is a direct consequence of the presence of this particular potential.

### 4.3 Symmetries of the composite dipole

The dipole Lagrangian (4.7) has the following symmetries and corresponding Noether charges

$$\delta X_i = \omega_{ij} X_j \qquad \boldsymbol{J} = \boldsymbol{z} \times \boldsymbol{\pi} + \boldsymbol{d} \times \boldsymbol{\sigma} \tag{4.19a}$$

$$\delta z_i = c_i \qquad \boldsymbol{P} = \boldsymbol{\pi} \tag{4.19b}$$

$$\delta \sigma_i = -\alpha_i \qquad \boldsymbol{D} = \boldsymbol{d} \tag{4.19c}$$

$$\delta \pi_i = -\epsilon d_i\,, \delta \sigma_i = -\epsilon z_i \qquad Z = \boldsymbol{d} \cdot \boldsymbol{z} \tag{4.19d}$$

$$\delta X_i = \lambda \dot{X}_i \qquad H = \frac{1}{2m} (\boldsymbol{\pi}^T)^2\,, \tag{4.19e}$$

where $\omega_{ij} = -\omega_{ji}$ and $X_i$ denotes all canonical variables. Using the Poisson brackets (4.2) leads to the expected symmetry algebra (1.5)

$$\{J_i, J_j\} = -\varepsilon_{ijk} J_k \tag{4.20a}$$

$$\{J_i, P_j\} = \varepsilon_{ijk} P_k \tag{4.20b}$$

$$\{J_i, D_j\} = \varepsilon_{ijk} D_k \tag{4.20c}$$

$$\{P_i, Z\} = -D_i\,. \tag{4.20d}$$

[5]This is analog to, e.g., the gravitational potential between two massive galilean particles, where the non-interacting particles are independently translation invariant $\delta x_{1,2} = a_{1,2}$, but the potential $V(\|x_1 - x_2\|)$ is only invariant under translations by the same parameter $\delta x_{1,2} = a$.

In particular

$$\{P_i, D_j\} = 0 \qquad\qquad \{H, Z\} = 0 \tag{4.21}$$

where the first relation shows that the charge of the dipole is indeed zero and the second that this system does not make use of the possible central extension (2.5). Let us emphasise that the rotations and the corresponding conserved angular momentum are of the whole system.

Indeed due to the second term in the Hamiltonian

$$H = \frac{1}{2m}\left[\boldsymbol{\pi}^2 - (\hat{\boldsymbol{d}} \cdot \boldsymbol{\pi})^2\right] \tag{4.22}$$

there is a distinguished rotation symmetry $(\boldsymbol{z}, \boldsymbol{\pi}, \boldsymbol{d}) \mapsto (R\boldsymbol{z}, R\boldsymbol{\pi}, R\boldsymbol{d})$ which preserves the dipole moment $R\hat{\boldsymbol{d}} = \hat{\boldsymbol{d}}$. This symmetry leads to the conserved longitudinal component of the angular momentum $L_i = P^L_{ij} J_j$. Additionally the dipoles also have a galilean boost symmetry in the transverse direction with a conserved center of mass, where both are explicitly given by

$$\boldsymbol{L} = \boldsymbol{z}^T \times \boldsymbol{\pi}^T \qquad\qquad \boldsymbol{K} = m\boldsymbol{z}^T - t\boldsymbol{\pi}^T\,. \tag{4.23}$$

Using these symmetries we find that the transverse part leads to a Bargmann-like central extension of the transversal Galilei algebra

$$\{L_i, P_j\} = \varepsilon_{ikl} P_{kj} P^T_l \tag{4.24a}$$

$$\{L_i, K_j\} = \varepsilon_{ikl} P_{kj} K_l \tag{4.24b}$$

$$\{K_i, P_j\} = m P_{ij} \tag{4.24c}$$

$$\{K_i, H\} = P^T_i\,. \tag{4.24d}$$

where the projector $P_{ij}$ takes the role of the usual Kronecker delta. This agrees with the intuition that the dipole can move like a massive galilean particle in the plane transverse to the dipole moment. Due to the presence of the projectors this is a Poisson, but no Lie algebra.

In the longitudinal direction there is no rotation symmetry, but

$$\boldsymbol{C} = m\boldsymbol{z}^L \qquad\qquad \boldsymbol{\sigma}^L \tag{4.25}$$

are conserved. In particular the first quantity can be seen as a carrollian center of mass leading to a Carroll-like algebra

$$\{C_i, P_j\} = m P^L_{ij} \tag{4.26a}$$

$$\{C_i, H\} = 0 \tag{4.26b}$$

$$\{D_i, \sigma^L_j\} = P^L_{ij}\,. \tag{4.26c}$$

This agrees with the intuition that the dipole is unable to move in the longitudinal direction.

Let us remark that there is also an action of the rotations $J_i$ on the longitudinal and transverse charges, but as is geometrically clear they do not leave their respective subspaces invariant, e.g., $\{J_i, z^T_j\} = \varepsilon_{ikl} P_{jl} z_k$ and $\{J_i, z^L_j\} = \varepsilon_{ikl} P^L_{jl} z_k$.

### 4.3.1 Reduced action

We can construct a reduced, but inequivalent, action principle by solving the equation of motion for the dipole moment $\dot{\boldsymbol{d}} = \boldsymbol{0}$, i.e., fixing the dipole moment to a constant value. This leads to $\boldsymbol{d}$ being a background value which we do not vary. The decomposition into longitudinal and transversal parts can then be done without using the equations of motion, i.e., off-shell. We can then integrate out $\boldsymbol{\pi}^T$ to obtain

$$L_{\boldsymbol{d}}\left[\boldsymbol{z}^T, \boldsymbol{z}^L, \boldsymbol{\pi}^L\right] = \frac{1}{2}m\|\dot{\boldsymbol{z}}^T\|^2 + \boldsymbol{\pi}^L \cdot \dot{\boldsymbol{z}}^L \,, \tag{4.27}$$

which leads directly to (4.10). This Lagrangian is the sum of a transverse galilean and a longitudinal carrollian particle action, thus manifesting the mixed galilean and carrollian symmetries we discussed in section 4.3.

However, the dipole conservation is not a consequence of the variational principle and the symmetries (4.19), in particular global rotation symmetries are absent.

## 4.4 Coupling of the dipole to a traceless fracton gauge field

Let us consider the coupling of the dipole (4.7) to the fracton traceless gauge field $(\phi, A_{ij})$ introduced in [23]. The interaction is described by the following Lagrangian

$$L_{\text{int}} = \int dt d^3x \left(\phi\rho - A_{ij}J^{ij}\right) . \tag{4.28}$$

Here $\phi$ represents the scalar potential, while $A_{ij}$ is the symmetric and traceless tensor potential. The charge and current densities of the dipole are given by:

$$\rho\left(t, \boldsymbol{x}\right) = -d^i\partial_i\delta^3\left(\boldsymbol{x} - \boldsymbol{z}\left(t\right)\right) \qquad\qquad J^{ij} = -d^{\langle i}\,\dot{z}^{j\rangle}\delta^3\left(\boldsymbol{x} - \boldsymbol{z}\left(t\right)\right) . \tag{4.29}$$

In these expressions, the brackets $\langle\cdots\rangle$ denote the symmetric and traceless part of a rank two tensor, i.e., $X_{\langle ij\rangle} = \frac{1}{2}\left(X_{ij} + X_{ji}\right) - \frac{1}{3}\delta_{ij}X^k{}_k$. We consider only the traceless part of $J^{ij}$ because this is the only part that couples to the traceless $A_{ij}$.

The total Lagrangian of the system is then given by

$$L = \pi_i\dot{z}^i + \sigma_i\dot{d}^i - \frac{1}{2m}P_{ij}\pi^i\pi^j + d^i\partial_i\phi(t, \boldsymbol{z}(t)) + A_{ij}(t, \boldsymbol{z}(t))d^{\langle i}\,\dot{z}^{j\rangle} \,, \tag{4.30}$$

which leads to the following equations of motion:

$$\dot{z}^i = \frac{1}{m}P^{ij}\pi_j \tag{4.31a}$$

$$\dot{\pi}_i = d^j\left(E_{ij} + \frac{1}{3}\delta_{ij}\partial^2\phi\right) + \left(F_{ijk} + \delta_{k[i}\partial^l A_{j]l}\right)\dot{z}^j d^k \tag{4.31b}$$

$$\dot{d}^i = 0 \tag{4.31c}$$

$$\dot{\sigma}_i = \frac{\hat{\boldsymbol{d}} \cdot \boldsymbol{\pi}}{md}P_{ij}\pi^j + \partial_i\phi + A_{ij}\dot{z}^j \,. \tag{4.31d}$$

Here,

$$E_{ij} := \partial_i\partial_j\phi - \frac{1}{3}\delta_{ij}\partial^2\phi - \dot{A}_{ij} \tag{4.32a}$$

$$F_{ijk} := \partial_i A_{jk} - \partial_j A_{ik} - \frac{1}{2}\delta_{ik}\partial^l A_{jl} + \frac{1}{2}\delta_{jk}\partial^l A_{il} \tag{4.32b}$$

denote the electric and magnetic fields, respectively, which are gauge-invariant quantities.

Remarkably, by taking the time derivative of (4.31a) and using (4.31b) and (4.31c), one obtains

$$m\ddot{z}^i = P^{ij} d^k \left( E_{jk} + F_{jlk} \dot{z}^l \right) , \tag{4.33}$$

which corresponds precisely to the Lorentz force found by Pretko [23] (where however it did not derive from an action principle). Like in the previous reference, one can also introduce the dualised magnetic field $B_{ij} := \frac{1}{2} \varepsilon_i{}^{pq} F_{pqj} \Leftrightarrow F_{ijk} = \varepsilon_{ij}{}^p B_{pk}$ which, like the electric field $E_{ij}$ is also gauge-invariant, symmetric and traceless.[6] Using this new quantity, the Lorentz force reads

$$m\ddot{z}^i = P^{ij} d^k \left( E_{jk} + \varepsilon_{jl}{}^p \dot{z}^l B_{pk} \right) = d \left( \boldsymbol{E}_{\text{eff}} + \dot{\boldsymbol{z}} \times \boldsymbol{B}_{\text{eff}} \right)^i . \tag{4.34}$$

where $E^i_{\text{eff}} = P^{ij} \hat{d}^k E_{jk}$ and $B^i_{\text{eff}} = P^{ij} \hat{d}^k B_{jk}$. Due to the presence of the projector $P^{ij}$, the velocity can be non-vanishing only in the direction transverse to the dipole moment. Indeed, contracting (4.31a) with $d^i$ yields $\dot{\boldsymbol{z}}^L = 0$.

Let us emphasise that the Lagrangian (4.30) and the equations of motion (4.31) are invariant under the following gauge transformations[7]

$$\delta_\Lambda \phi = \partial_t \Lambda \qquad \delta_\Lambda A_{ij} = \partial_i \partial_j \Lambda - \frac{1}{3} \delta_{ij} \partial^2 \Lambda \qquad \delta_\Lambda \pi^i = \frac{1}{3} d^i \partial^2 \Lambda \qquad \delta_\Lambda \sigma_i = \partial_i \Lambda . \tag{4.35}$$

In particular, by selecting $\Lambda = -\frac{1}{2} \epsilon z_j z^j - \alpha_j z^j$, one obtains

$$\delta_\Lambda \pi^i = -\epsilon d^i \qquad\qquad \delta_\Lambda \sigma_i = -\epsilon z_i - \alpha_i , \tag{4.36}$$

which are precisely the transformations of $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ under quadrupole and dipole transformations. Hence, the global symmetries (4.3) can be regarded as "large gauge transformations."

### 4.5 First quantisation of the composite dipole with fracton gauge field

In this subsection, we shall perform the first quantisation of the composite dipole interacting with an external fracton gauge field. To this end, we shall formulate the Lagrangian (4.30) in canonical form, making the invariance under arbitrary time reparametrisations explicit. Following Dirac's approach, we shall then impose the condition that the wave function be annihilated by the constraint generating time reparametrisations, with the momenta substituted by their corresponding quantum operators.

Since the interaction term depends explicitly on the velocity of the dipole, the field $\pi_i$ is no longer the canonical conjugate to the position. Instead, it is given as

$$p_i := \pi_i + d^j A_{ij} , \tag{4.37}$$

which still satisfies the canonical Poisson bracket $\{z^i, p_j\} = \delta^i_j$. This situation is similar to that of a particle coupled to an external electromagnetic field.

[6] Our definition for the magnetic field $B^{\text{here}}_{ij} = \varepsilon_i{}^{pq} (\partial_p A_{qj} - \frac{1}{2} \delta_{jp} \partial^k A_{qk})$ differs slightly from Pretko's magnetic field $B^{\text{there}}_{ij} = \varepsilon_i{}^{pq} \partial_p A_{qj}$, which is not symmetric. This has no bearing on the expression of the Lorentz force however, since the extra terms vanish by virtue of the equations of motion.

[7] While checking the gauge invariance, the following identity is useful $\frac{d}{dt} \Lambda(t, \boldsymbol{z}) = \partial_t \Lambda(t, \boldsymbol{z}) + \dot{z}^i \partial_i \Lambda(t, \boldsymbol{z})$.



JHEP11(2025)030

In terms of $p_i$, the Lagrangian (4.30) reads

$$L = p_i \dot{z}^i + \sigma_i \dot{d}^i - \frac{1}{2m} P^{ij} \left(p_i - d^k A_{ik}\right) \left(p_j - d^l A_{il}\right) + d^i \partial_i \phi \,. \tag{4.38}$$

To express the action in a form that is manifestly invariant under time reparametrisations, we must treat the time $t$ as a canonical variable and introduce a conjugate momentum $p_t$, such that $\{t, p_t\} = -1$. Then, we can write

$$L = -p_t \frac{d}{d\tau} t + p_i \frac{d}{d\tau} z^i + \sigma_i \frac{d}{d\tau} d^i - N \left(p_t + d^i \partial_i \phi - \frac{1}{2m} P^{ij} \left(p_i - d^k A_{ik}\right) \left(p_j - d^l A_{il}\right)\right) \,, \tag{4.39}$$

where $\tau$ denotes the proper time parameter, and $N$ is the Lagrange multiplier that enforces the constraint

$$p_t + d^i \partial_i \phi - \frac{1}{2m} P^{ij} \left(p_i - d^k A_{ik}\right) \left(p_j - d^l A_{il}\right) \approx 0 \,, \tag{4.40}$$

which generates arbitrary time reparametrisations.

The wave equation is obtained by requiring that the constraint annihilates the wave function $\psi\,(t, \boldsymbol{z}, \boldsymbol{d})$, where the momenta are replaced by the quantum operators

$$p_t = i \frac{\partial}{\partial t} \qquad\qquad p_i = -i \frac{\partial}{\partial z^i} \qquad\qquad \sigma_i = -i \frac{\partial}{\partial d^i} \,. \tag{4.41}$$

Thus, one obtains the following Schrödinger-like wave equation

$$\left[i \frac{\partial}{\partial t} + d^i \partial_i \phi - \frac{1}{2m} P^{ij} \left(i \frac{\partial}{\partial z^i} + d^k A_{ik}\right) \left(i \frac{\partial}{\partial z^j} + d^l A_{jl}\right)\right] \psi\,(t, \boldsymbol{z}, \boldsymbol{d}) = 0 \,. \tag{4.42}$$

Note that, since the field $\boldsymbol{\sigma}$ does not appear explicitly in the constraint (4.40), there are no derivatives with respect to the dipole moment in the wave equation. Consequently, the dipole moment $\boldsymbol{d}$ can be treated as a fixed parameter within the wave equation.

The above equation of motion arises from variation of the following Lagrangian density

$$\mathcal{L}[\psi, \psi^*] = i\psi^* \mathcal{D}_t \psi + \frac{1}{2m} \psi^* P^{ij} \mathcal{D}_i \mathcal{D}_j \psi \tag{4.43a}$$

$$= i\psi^* \partial_t \psi + \frac{1}{2m} \psi^* P^{ij} \partial_i \partial_j \psi + d^i \psi^* \psi \partial_i \phi + \frac{i d^j}{2m} P^{ik} \left(\psi \partial_k \psi^* - \psi^* \partial_k \psi\right) A_{ij} - \frac{1}{2m} d^k d^l P^{ij} A_{ik} A_{jl} \psi^* \psi \,, \tag{4.43b}$$

where we introduced the covariant derivatives

$$\mathcal{D}_t := \partial_t - i d^i \partial_i \phi \qquad\qquad \mathcal{D}_i := \partial_i - i A_{ik} d^k \,, \tag{4.44}$$

with $\partial_t := \partial/\partial t$ and $\partial_i := \partial/\partial z^i$.

This Lagrangian is invariant under gauge transformations of the form

$$\delta_\Lambda \phi = \partial_t \Lambda \qquad\qquad \delta_\Lambda A_{ij} = \partial_i \partial_j \Lambda - \frac{1}{3} \delta_{ij} \partial^2 \Lambda \qquad\qquad \delta_\Lambda \psi = i d^i \left(\partial_i \Lambda\right) \psi \,. \tag{4.45}$$

In particular, the covariant derivatives transform as

$$\delta_\Lambda(\mathcal{D}_t \psi) = i d^j \left(\partial_j \Lambda\right) \mathcal{D}_t \psi \qquad\qquad \delta_\Lambda(\mathcal{D}_i \psi) = \left[i d^j \left(\partial_j \Lambda\right) \mathcal{D}_i - \frac{1}{3} d_i \partial^2 \Lambda\right] \psi \,. \tag{4.46}$$



JHEP11(2025)030

Additionally, it is invariant under a global $U(1)$ transformation $\delta\psi = i\alpha\psi$, which is associated with the electric charge.

In the specific case of large gauge transformations of the form $\Lambda = -\frac{1}{2}\epsilon \|\boldsymbol{z}\|^2 - \boldsymbol{\alpha}\cdot\boldsymbol{z}$, where $\epsilon$ is the parameter of the trace charge and $\boldsymbol{\alpha}$ represents the parameter for dipole transformations, the wave function transforms as follows:

$$\delta\psi = -i\boldsymbol{d}\cdot(\boldsymbol{\alpha}+\epsilon\boldsymbol{z})\,\psi\,. \tag{4.47}$$

The free Schrödinger-like Lagrangian

$$\mathcal{L}_{\text{free}} = i\psi^*\partial_t\psi + \frac{1}{2m}\psi^* P^{ij}\partial_i\partial_j\psi\,, \tag{4.48}$$

is invariant under the global transformations in (4.47).

The dispersion relation associated with the free Lagrangian (4.48) takes the form

$$\omega = \frac{1}{2m}P^{ij}k_ik_j\,. \tag{4.49}$$

Here, $\omega$ represents the frequency, and $k_i$ denotes the momentum of the excitation. It is important to note that the components of the momentum $\boldsymbol{k}$ in the direction of the dipole moment $\hat{\boldsymbol{d}}$ do not contribute to the energy.

In $2+1$ dimensions, the Lagrangian density (4.43) was previously derived in [24, 25] (see also [7, section III.B.3]), in the context of the study of the fracton/elasticity duality. Their derivation of the Lagrangian followed a different approach from ours, where the effective dipole dynamics was obtained directly from a field theory, rather than through the quantisation of the classical particle-like model for a dipole as discussed in section 4.4.

### 4.6 Symmetries of the free planon field theory

Let us determine the global conserved charges of the Lagrangian (4.48) that describes a free quantum dipole. The Noether current associated with the global transformations (4.47) is given by

$$j^0 = \boldsymbol{d}\cdot(\boldsymbol{\alpha}+\epsilon\boldsymbol{z})\,\psi^*\psi \qquad j^i = -\frac{i}{2m}\boldsymbol{d}\cdot(\boldsymbol{\alpha}+\epsilon\boldsymbol{z})\,P^{ij}\left(\psi^*\partial_j\psi - \psi\partial_j\psi^*\right)\,, \tag{4.50}$$

where the current satisfies the continuity equation $\partial_t j^0 + \partial_i j^i = 0$. As a consequence, the corresponding conserved charges are given by

$$Q\left[\boldsymbol{\alpha},\epsilon\right] = \int d^3\boldsymbol{z}\, j^0 = \boldsymbol{\alpha}\cdot\boldsymbol{D} + \epsilon\, Z\,, \tag{4.51}$$

where the dipole moment $\boldsymbol{D}$ and the trace charge $Z$ are given by

$$\boldsymbol{D} = \boldsymbol{d}\int d^3\boldsymbol{z}\,\psi^*\psi = \boldsymbol{d} \qquad Z = \int d^3\boldsymbol{z}\,\left(\boldsymbol{d}\cdot\boldsymbol{z}\right)\psi^*\psi\,. \tag{4.52}$$

Note that we have applied the standard normalisation condition for the wave function $\int d^3\boldsymbol{z}\,\psi^*\psi = 1$.

A natural question is whether these conserved quantities can be derived from a continuity equation of the form typically found in fracton theories. Specifically, for a planon symmetry, one expects an equation of the form

$$\partial_t \rho + \partial_i \partial_j J^{ij} = 0, \tag{4.53}$$

where $\rho$ represents the electric charge density, and $J^{ij}$ is a symmetric, traceless tensor describing the dipole current. The answer is positive for a charge and current density of the form

$$\rho = -d^i \partial_i \left(\psi^* \psi\right) \qquad\qquad J^{ij} = \frac{i}{2m} d^{(i} P^{j)k} \left(\psi^* \partial_k \psi - \psi \partial_k \psi^*\right). \tag{4.54}$$

Note that this expression for the charge density resembles that of a classical dipole, except that the $\delta$-function specifying the position of the classical dipole is replaced by the probability density of the quantum dipole. In particular, the total electric charge vanishes

$$Q = \int d^3\boldsymbol{z}\, \rho = -\int d^3\boldsymbol{z}\, \partial_j \left(d^j \psi^* \psi\right) = 0\,, \tag{4.55}$$

which is consistent with the fact that a dipole is electrically neutral.

Additionally, this quantum system retains the mixed Carroll-Galilei boosts present in its classical counterpart discussed in section 4.3. However, a key difference arises: the commutator between a longitudinal Carroll boost and a transverse galilean boost no longer vanishes.

Under infinitesimal transverse Galilei boosts of the form $\delta x^i = -P^{ij} v_j t$ the wave function transforms according to

$$\delta\psi = P^{ij} v_i \left(t \partial_j \psi - i m z_j \psi\right), \tag{4.56}$$

where the second term corresponds to a change in the phase. Applying Noether's theorem, the transformation above yields the following conserved quantity associated with galilean boosts

$$K_i = \int d^3\boldsymbol{z}\, i P^{ij} \left(t \psi^* \partial_j \psi - i m z_j \psi^* \psi\right). \tag{4.57}$$

Under infinitesimal longitudinal Carroll boosts of the form $\delta t = -P_L^{ij} v_i z_j$, the wave function transforms according to

$$\delta\psi = P_L^{ij} v_i z_j \dot{\psi}\,. \tag{4.58}$$

Therefore, the Noether charge corresponding to longitudinal Carroll boosts is given by

$$C_i = \int d^3\boldsymbol{z}\, \frac{1}{2m} P_L^{ij} P^{kl} z_j \left(\partial_k \psi^*\right) \left(\partial_l \psi\right). \tag{4.59}$$

Using the Poisson bracket $\{\psi(z), \psi^*(z')\} = \frac{1}{i}\delta(z - z')$, one finds

$$\{K_i, C_j\} = X_{ij}, \tag{4.60}$$

where

$$X_{ij} := -i P^{il} P_L^{jm} \int d^3\boldsymbol{z}\, z_m \psi^* \partial_l \psi, \tag{4.61}$$

defines a new conserved quantity.



JHEP11(2025)030

On the other hand, the generators associated with spacetime translations take the expected form

$$H = -\int d^3\boldsymbol{z}\, \frac{1}{2m} P^{ij} \psi^* \partial_i \partial_j \psi \qquad\qquad P_i = -\int d^3\boldsymbol{z}\, i\psi^* \partial_i \psi\,. \tag{4.62}$$

The case of spatial rotations differs from the particle case studied in section 4.3, as the dipole moment $d_i$ is now fixed. In other words, it is treated as a background field that is not varied in the action, whereas in the particle case it corresponds to a canonical variable. As a result, its presence breaks the full rotational symmetry down to the subgroup that leaves the dipole direction invariant, i.e., those satisfying $R\hat{\boldsymbol{d}} = \hat{\boldsymbol{d}}$. Consequently, the generator of rotations around $\hat{\boldsymbol{d}}$ is given by

$$L_i = -iP_L^{il} \varepsilon_{ljk} \int d^3\boldsymbol{z}\, z^j \psi^* \partial_k \psi\,. \tag{4.63}$$

To restore rotational symmetry around all axes, it is necessary to allow for the rotation of the entire system, including the dipole moment. One possibility is to consider $\boldsymbol{d}$ as an additional variable over which we integrate the Lagrangian (4.43), i.e., the action is $\int \mathcal{L}[\psi, \psi^*] dt d^3\boldsymbol{z} d^3\boldsymbol{d}$. Indeed, according to the quantisation conditions (4.41) and (4.42) the wavefunction should depend on all of these coordinates. In this extended configuration space the dipole moment will transform under rotations and rotational symmetry is restored. This is reminiscent of systems with spin, where we sum over all spins, but for the dipoles it is an integral rather than a sum. It is an interesting perspective to consider on this extended space, but henceforth we will again work with fixed $\boldsymbol{d}$.

Similar to the standard Schrödinger equation, this model also possesses a dilatation symmetry

$$\delta\psi = \lambda_1 \left( 2t\partial_t \psi + z_k \partial_k \psi + \frac{3}{2}\psi \right), \tag{4.64}$$

whose charge is given by

$$\Delta_1 = \int d^3\boldsymbol{z} \left( -\frac{t}{m} P^{ij} \psi^* \partial_i \partial_j \psi + i z_k \psi^* \partial_k \psi + \frac{3i}{2} \psi^* \psi \right). \tag{4.65}$$

Interestingly, this system also possesses a second type of dilatation symmetry

$$\delta\psi = \lambda_2 \left( 2t\partial_t \psi + P^{kl} z_k \partial_l \psi + \psi \right), \tag{4.66}$$

with a conserved charge given by

$$\Delta_2 = \int d^3\boldsymbol{z} \left[ -\frac{t}{m} P^{ij} \psi^* \partial_i \partial_j \psi + i P^{kl} z_k \psi^* \partial_l \psi + i\psi^* \psi \right]. \tag{4.67}$$

This does not constitute a complete analysis of all possible symmetries of the Lagrangian (4.48). However, the preceding results already illustrate the richness of this class of models. Once the system is coupled to a gauge field, most of these symmetries are broken. In particular, transverse galilean and longitudinal Carroll boosts no longer define symmetries of the fractonic gauge field theory, due to its inherently aristotelian structure.



JHEP11(2025)030

## 5 Quantum planon particles

The next stage in the study of the planon group is the construction of its unitary irreducible (projective) representations, which we may identify with the elementary quantum particles of the planon group. These are again honest unitary irreducible representations of the centrally extended planon group. As we saw in section 2, this group is a semidirect product $G = B \ltimes A$, with $B$ the Bargmann group and $A$ the two-dimensional abelian group with Lie algebra $\mathfrak{a} = \langle H, W\rangle$. In principle, we may apply the method of Mackey to classify its UIRs, as was used for fractons in [26], for instance, *provided* that the semidirect product is regular (see, e.g., [37, Ch. 17.1]). This is proved in appendix C.

Since $A$ is abelian, its unitary irreducible representations are one-dimensional and the unitary dual $\widehat{A} \cong \mathbb{R}^2$, associating with $(E, c) \in \mathbb{R}^2$ the character $\chi : A \to \mathrm{U}(1)$ defined by $\chi(\exp(tH + wW)) = e^{i(Et+cw)}$. The action of $B$ on $A$ induces an action on $\widehat{A}$. Only the generator $Z$ acts nontrivially and we see that

$$\exp(\tilde{\varphi}Z) \cdot (E, c) = (E + c\tilde{\varphi}, c), \tag{5.1}$$

following on from the coadjoint action $\mathrm{ad}^*_Z H = -W$. The unitary dual $\widehat{A}$ decomposes under the action of $B$ into the following orbits:

- point-like orbits $\{(E, 0)\}$ for every $E \in \mathbb{R}$;
- and one-dimensional orbits $\mathbb{R} \times \{c\}$ for every $c \neq 0$.

The point-like orbits are stabilised by the full Bargmann group $B$, whereas a point $(E, c)$, with $c \neq 0$, in a one-dimensional orbit is stabilised by the subgroup whose Lie algebra is the span of $L_{ab}, P_a, D_a, Q$ and is isomorphic to the Carroll algebra. The action of $G$ on the one-dimensional orbits is by translations, and therefore the standard Lebesgue measure on the real line is invariant.

We therefore simply induce UIRs of $G$ from characters $\chi \in \widehat{A}$ and a UIR of its stabiliser subgroup $G_\chi$. We must distinguish between the two kinds of characters.

### 5.1 UIRs for characters in point-like orbits

For $\chi$ a character in a point-like orbit, $G_\chi = B$ and hence the corresponding UIRs of $G$ are in bijective correspondence with the UIRs of the Bargmann group, which can be read off from [1, table 6] (for $n = 3$, which we tacitly assume), after translating the notation from Bargmann to planon language. There are five classes of such UIRs, which we now summarise. We use the notation $g := \mathbf{g}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R, s, w) = \mathbf{b}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R) \exp(sH + wW)$.

#### 5.1.1 $I(\ell, \tilde{q}, E)$

These are finite-dimensional UIRs with Hilbert space $\mathcal{H} = V_\ell$, the complex spin-$\ell$ UIR of Spin(3). We let $\rho_\ell$ denote the representation map $\rho_\ell : \mathrm{Spin}(3) \to \mathrm{GL}(V_\ell)$. The representation $U : G \to U(\mathcal{H})$ is given by

$$U(g)\psi = e^{isE} e^{i\tilde{q}\tilde{\varphi}} \rho_\ell(R)\psi. \tag{5.2}$$

### 5.1.2 $II(\ell, q, \tilde{q}, E)$

Here the Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^3, V_\ell)$ relative to the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{R}^3} d\mu(\boldsymbol{d}) \, \langle \psi_1(\boldsymbol{d}), \psi_2(\boldsymbol{d}) \rangle_{V_\ell} \,, \tag{5.3}$$

with $d\mu(\boldsymbol{d})$ the standard euclidean measure on $\mathbb{R}^3$. The representation $U : G \to U(\mathcal{H})$ is given by

$$\left(U(g)\psi\right)(\boldsymbol{d}) = e^{isE} e^{iq\varphi} e^{i\tilde{q}\tilde{\varphi}} e^{i\boldsymbol{\beta}\cdot\boldsymbol{d}} \rho_\ell(R) \psi(R^{-1}(\boldsymbol{d} + q\boldsymbol{a})). \tag{5.4}$$

### 5.1.3 $III(n, p, \tilde{q}, E)$

Here the Hilbert space is $\mathcal{H} = L^2(S^2, \mathscr{O}(-n))$, which we think of as complex-valued smooth (i.e., not necessarily holomorphic) functions $\psi(z)$ of a stereographic coordinate $z$ for $S^2$, which are square-integrable relative to the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{C}} \frac{2idz \wedge d\bar{z}}{(1 + |z|^2)^2} \overline{\psi_1(z)} \psi_2(z). \tag{5.5}$$

The representation $U : G \to U(\mathcal{H})$ is given by

$$\left(U(g)\psi\right)(z) = e^{isE} e^{i\tilde{q}\tilde{\varphi}} e^{i\boldsymbol{a}\cdot\boldsymbol{\pi}(z)} \left(\frac{\eta + \bar{\xi}z}{|\eta + \bar{\xi}z|}\right)^{-n} \psi\left(\frac{\bar{\eta}z - \xi}{\eta + \bar{\xi}z}\right), \tag{5.6}$$

where

$$R = \begin{pmatrix} \eta & \xi \\ -\bar{\xi} & \bar{\eta} \end{pmatrix} \tag{5.7}$$

and

$$\boldsymbol{\pi}(z) = \frac{p}{1 + |z|^2} \left(2\,\mathrm{Re}(z), 2\,\mathrm{Im}(z), |z|^2 - 1\right). \tag{5.8}$$

### 5.1.4 $IV(n, d, E)$

Here the Hilbert space is $\mathcal{H} = L^2(\mathbb{R} \times S^2, \mathscr{O}(-n))$, where the complex line bundle $\mathscr{O}(-n)$ on $S^2$ has been pulled back to $\mathbb{R} \times S^2$ via the cartesian projection $\mathbb{R} \times S^2 \to S^2$. As in the previous case we think of these as complex-valued smooth functions $\psi(u, z)$, where $z$ is a stereographic coordinate for $S^2$, which are square-integrable relative to the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{R}\times\mathbb{C}} \frac{2idu \wedge dz \wedge d\bar{z}}{(1 + |z|^2)^2} \overline{\psi_1(u, z)} \psi_2(u, z). \tag{5.9}$$

The representation $U : G \to U(\mathcal{H})$ is given by

$$\left(U(g)\psi\right)(u, z) = e^{isE} e^{-iu\tilde{\varphi}} e^{-i\boldsymbol{\beta}\cdot\boldsymbol{\delta}(z)} \left(\frac{\eta + \bar{\xi}z}{|\eta + \bar{\xi}z|}\right)^{-n} \psi\left(u - \boldsymbol{a}\cdot\boldsymbol{\delta}(z), \frac{\bar{\eta}z - \xi}{\eta + \bar{\xi}z}\right), \tag{5.10}$$

where $R$ is as in equation (5.7) and

$$\boldsymbol{\delta}(z) = \frac{d}{1 + |z|^2} \left(2\,\mathrm{Re}(z), 2\,\mathrm{Im}(z), |z|^2 - 1\right). \tag{5.11}$$



JHEP11(2025)030

#### 5.1.5 $V_{\pm}(\boldsymbol{d}, \boldsymbol{p}^{\perp}, \boldsymbol{E})$

Here the Hilbert space is $\mathcal{H} = \Pi_{\pm} L^2(\mathbb{R} \times S^3, \mathbb{C})$, where the projectors $\Pi_{\pm}$ are given by

$$(\Pi_{\pm}\psi)(u, S) = \frac{1}{2}(\psi(u, S) \pm \psi(u, -S)) \tag{5.12}$$

where $S \mapsto -S$ sends a point on $S^3$ to its antipodal point or, equivalently, if we identify $S^3$ with SU(2), then this is just multiplying the matrix $S$ by $-1$. These functions are square-integrable relative to the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{R}\times S^3} du \wedge d\mu(S) \overline{\psi_1(u, S)} \psi_2(u, S), \tag{5.13}$$

with $d\mu(S)$ a bi-invariant volume form on SU(2). The representation $U : G \to U(\mathcal{H})$ is given by

$$(U(g)\psi)(u, S) = e^{isE} e^{i\boldsymbol{a}\cdot S\boldsymbol{p}} e^{i\boldsymbol{\beta}\cdot S\boldsymbol{d}} e^{i(\tilde{\varphi}+u)\boldsymbol{a}\cdot S\boldsymbol{d}} \psi(u + \tilde{\varphi}, R^{-1}S), \tag{5.14}$$

where the action of $S \in \mathrm{SU}(2)$ on vectors is via the covering homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

### 5.2 UIRs for characters in one-dimensional orbits

For $\chi$ a character in a one-dimensional orbit, we obtain UIRs as square-integrable functions from the real line (the one-dimensional orbit of $\chi$) with values in a UIR of the Carroll group, which for $n = 3$ can be read off from [26, table 5]. Such characters are determined by a nonzero $c \in \mathbb{R}$ and we can choose as an orbit representative the character $\chi$ corresponding to $(0, c)$; that is,

$$\chi(\exp(sH + wW)) = e^{icw}. \tag{5.15}$$

We must choose $\sigma : \mathbb{R} \to G$ such that $\sigma(E) \cdot (0, c) = (E, c)$ and we take $\sigma(E) = \exp(\frac{E}{c}Z)$, which is well-defined since $c \neq 0$. The generic group element can be written as

$$g := \mathbf{g}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R, s, w) = e^{\tilde{\varphi}Z} \mathbf{c}(\varphi, \boldsymbol{\beta}, \boldsymbol{a}, R) \exp(sH + wW), \tag{5.16}$$

where $\mathbf{c}(\varphi, \boldsymbol{\beta}, \boldsymbol{a}, R) = e^{\varphi Q} e^{\boldsymbol{\beta}\cdot\boldsymbol{D}} e^{\boldsymbol{a}\cdot\boldsymbol{P}} R$ is a general element of the Carroll subgroup stabilising $\chi$. Let $G_\chi = C \ltimes A$ denote the stabiliser subgroup of $\chi$, where $C$ is the Carroll subgroup and $A$ the abelian subgroup generated by $H, W$. Let $\mathcal{H}$ be a Carroll UIR twisted by the UIR of $A$ defined by the character $\chi$ and let $F : G \to \mathcal{H}$ be a $G_\chi$-equivariant (Mackey) function; that is,

$$F(gh) = h^{-1} \cdot F(g) \qquad \forall g \in G, h \in G_\chi. \tag{5.17}$$

We define the corresponding section $\psi$ of the (trivial) vector bundle $\mathbb{R} \times \mathcal{H}$ over the real line by $\psi(E) = F(\sigma(E))$, which transforms under $g \in G$ as

$$(g \cdot \psi)(E) = F(g^{-1}\sigma(E)). \tag{5.18}$$

A calculation shows that

$$g^{-1}\sigma(E) = \sigma(E - c\tilde{\varphi}) \mathbf{c}\left(\boldsymbol{a}\cdot\boldsymbol{\beta} - \varphi - \frac{1}{2}\left(\frac{E}{c} - \tilde{\varphi}\right)\|\boldsymbol{a}\|^2, -R^{-1}(\boldsymbol{\beta} - \left(\frac{E}{c} - \tilde{\varphi}\right)\boldsymbol{a}), -R^{-1}\boldsymbol{a}, R^{-1}\right) \times e^{-sH} e^{-(w+t(\frac{E}{c}-\tilde{\varphi}))W}. \tag{5.19}$$

Inserting this into $F$ and using the $G_\chi$-equivariance, we arrive at

$$(g\cdot\psi)(E) = e^{i(cw+s(E-c\tilde{\varphi}))}U\left(\mathbf{c}\left(\varphi-\frac{1}{2}\left(\frac{E}{c}-\tilde{\varphi}\right)\|\boldsymbol{a}\|^2, \boldsymbol{\beta}-\left(\frac{E}{c}-\tilde{\varphi}\right)\boldsymbol{a}, \boldsymbol{a}, R\right)\right)\cdot\psi(E-c\tilde{\varphi}), \tag{5.20}$$

which is unitary relative to the inner product

$$(\psi_1,\psi_2) = \int_{\mathbb{R}} dE\, \langle\psi_1(E),\psi_2(E)\rangle_{\mathcal{H}}, \tag{5.21}$$

where $\langle -,-\rangle_{\mathcal{H}}$ is the inner product on the UIR $\mathcal{H}$. It is now a simple matter to go through the Carroll UIRs in [26, section3.4] and translate the explicit representations of the Carroll group to the planon language. As was the case already with the Carroll and Bargmann groups, some of the induced representations of $G$ thus constructed appear at face value to be carried by sections of infinite-rank Hilbert bundles over the character orbits. But in the same way that it was possible in the Carroll and Bargmann cases to view such representations as carried by sections of finite-rank vector bundles over an auxiliary space fibering over the character orbit, we can do the same here.

#### 5.2.1 $VI(\ell, c)$

These are induced from the finite-dimensional representations of the Carroll group in which the rotations act on the complex spin-$\ell$ representation $V_\ell$ and the other generators act trivially. Hence $\mathcal{H} = L^2(\mathbb{R}, V_\ell)$ relative to the inner product

$$(\psi_1,\psi_2) = \int_{\mathbb{R}} dE\, \langle\psi_1(E),\psi_2(E)\rangle_{V_\ell} \tag{5.22}$$

and the generic element $g := \mathbf{g}(\varphi,\tilde{\varphi},\boldsymbol{\beta},\boldsymbol{a},R,s,w)\in G$ acts on $\mathcal{H}$ as

$$(U(g)\psi)(E) = e^{i(cw+s(E-c\tilde{\varphi}))}\rho_\ell(R)\psi(E-c\tilde{\varphi}). \tag{5.23}$$

#### 5.2.2 $VII(\ell, q, c)$

The inducing representation is carried by $L^2(\mathbb{R}^3, V_\ell)$ relative to the inner product

$$(\psi_1,\psi_2) = \int_{\mathbb{R}^3} d\mu(\boldsymbol{d})\, \langle\psi_1(\boldsymbol{d}),\psi_2(\boldsymbol{d})\rangle_{V_\ell}, \tag{5.24}$$

with $d\mu(\boldsymbol{d})$ the standard euclidean volume form on $\mathbb{R}^3$. This means that the induced representation can be reformulated as carried by $\mathcal{H} = L^2(\mathbb{R}^4, V_\ell)$ relative to the inner product

$$(\psi_1,\psi_2) = \int_{\mathbb{R}^4} dE\wedge d\mu(\boldsymbol{d})\, \langle\psi_1(E,\boldsymbol{d}),\psi_2(E,\boldsymbol{d})\rangle_{V_\ell}, \tag{5.25}$$

with the generic group element $g\in G$ acting on $\mathcal{H}$ via

$$\begin{aligned}(U(g)\psi)(E,\boldsymbol{d}) = {}& e^{i(cw+s(E-c\tilde{\varphi})+q(\varphi-\frac{1}{2c}(E-c\tilde{\varphi})\|\boldsymbol{a}\|^2)+(\boldsymbol{\beta}-\frac{1}{c}(E-c\tilde{\varphi})\boldsymbol{a})\cdot\boldsymbol{d})}\\ &\times\rho_\ell(R)\psi(E-c\tilde{\varphi}, R^{-1}(\boldsymbol{d}+q\boldsymbol{a})).\end{aligned} \tag{5.26}$$

### 5.2.3 $\boldsymbol{VIII(n, d, c)}$

The inducing representation is carried by $L^2(S^2, \mathscr{O}(-n))$ which we describe as in section 5.1.4 in terms of smooth functions of a stereographic coordinate $z$. We may reformulate this UIR as carried by $L^2(\mathbb{R} \times S^2, \mathscr{O}(-n))$, where $\mathscr{O}(-n)$ has been pulled-back to $\mathbb{R} \times S^2$ via the cartesian projection, relative to the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{R}\times\mathbb{C}} \frac{dE \wedge 2idz \wedge d\bar{z}}{(1+|z|^2)^2} \overline{\psi_1(E, z)} \psi_2(E, z) \tag{5.27}$$

and the action of the generic element $g \in G$ is given by

$$\begin{aligned}(U(g)\psi)(E, z) = & e^{i(cw+s(E-c\tilde{\varphi}))} e^{i(\beta - \frac{1}{c}(E-c\tilde{\varphi})\boldsymbol{a})\cdot \boldsymbol{\delta}(z)} \\ & \times \left(\frac{\eta + \bar{\xi}z}{|\eta + \bar{\xi}z|}\right)^{-n} \psi\left(E - c\tilde{\varphi}, \frac{\bar{\eta}z - \xi}{\eta + \bar{\xi}z}\right),\end{aligned} \tag{5.28}$$

with $R$ as in equation (5.7) and $\boldsymbol{\delta}(z)$ as in equation (5.11).

### 5.2.4 $\boldsymbol{IX(n, p, c)}$

This is very similar to the above case, with the inducing UIRs related by a Carroll automorphism. We reformulate the UIR again as carried by $L^2(\mathbb{R} \times S^2, \mathscr{O}(-n))$, relative to the inner product in equation (5.27) and now the action of the generic element $g \in G$ is given by

$$(U(g)\psi)(E, z) = e^{i(cw+s(E-c\tilde{\varphi}))} e^{i\boldsymbol{a}\cdot\boldsymbol{\pi}(z)} \left(\frac{\eta + \bar{\xi}z}{|\eta + \bar{\xi}z|}\right)^{-n} \psi\left(E - c\tilde{\varphi}, \frac{\bar{\eta}z - \xi}{\eta + \bar{\xi}z}\right), \tag{5.29}$$

with $R$ as in equation (5.7) and $\boldsymbol{\pi}(z)$ as in equation (5.8).

### 5.2.5 $\boldsymbol{X_\pm(n, d, p, c)}$

This representation is carried by the same Hilbert space as in the previous two cases: namely, $L^2(\mathbb{R} \times S^2, \mathscr{O}(-n))$ relative to the usual inner product given by equation (5.27). In these UIRs, the vectors $\boldsymbol{p}$ and $\boldsymbol{d}$ are either parallel (corresponding to the $+$ sign) or anti-parallel (corresponding to the $-$ sign). The action of the generic $g \in G$ is given by

$$\begin{aligned}(U(g)\psi)(E, z) = & e^{i(cw+s(E-c\tilde{\varphi}))} e^{i(\beta - \frac{1}{c}(E-c\tilde{\varphi})\boldsymbol{a})\cdot \boldsymbol{\delta}(z) \pm \boldsymbol{a}\cdot\boldsymbol{\pi}(z))} \\ & \times \left(\frac{\eta + \bar{\xi}z}{|\eta + \bar{\xi}z|}\right)^{-n} \psi\left(E - c\tilde{\varphi}, \frac{\bar{\eta}z - \xi}{\eta + \bar{\xi}z}\right),\end{aligned} \tag{5.30}$$

where $R$ as in equation (5.7), $\boldsymbol{\pi}(z)$ as in equation (5.8) and $\boldsymbol{\delta}(z)$ as in equation (5.11).

### 5.2.6 $\boldsymbol{XI_\pm(d, p, \theta, c)}$

Finally, we have the representation for which $\boldsymbol{d} = (0, 0, d)$ and $\boldsymbol{p} = p(\sin\theta, 0, \cos\theta)$. This is carried by $L^2_\pm(\mathbb{R} \times S^3, \mathbb{C})$, where $L^2_\pm(\mathbb{R} \times S^3, \mathbb{C}) \subset L^2(\mathbb{R} \times S^3, \mathbb{C})$ is the closed subspace consisting of square-integrable functions $\psi$ such that $\psi(E, -S) = \pm\psi(E, S)$, where $E \in \mathbb{R}$ and $S \in S^3$, but thought of as a matrix in SU(2). The inner product is given by

$$(\psi_1, \psi_2) = \int_{\mathbb{R}\times S^3} dE \wedge d\mu(S) \overline{\psi_1(E, S)} \psi_2(E, S), \tag{5.31}$$

| UIR Label | Orbit |
|---|---|
| I$(0,\tilde{q},E)$ | $0(\tilde{q},E)$ |
| I$(\ell,\tilde{q},E)$ | $2(\ell,\tilde{q},E)$ |
| II$(0,q,\tilde{q},E)$ | $6(q,\tilde{q},E)$ |
| II$(\ell,q,\tilde{q},E)$ | $8(\ell,q,\tilde{q},E)$ |
| III$(n,p,\tilde{q},E)$ | $4(h=n,p,\tilde{q},E)$ |
| IV$(n,d,E)$ | $6'(h=n,d,E)$ |
| V$_\pm(d,p^\perp,E)$ | $8'(d,p^\perp,E)$ |
| VI$(0,c)$ | $2'(c)$ |
| VI$(\ell,c)$ | $4'(\ell,c)$ |
| VII$(0,q,c)$ | $8'''(q,c)$ |
| VII$(\ell,q,c)$ | $10(\ell,q,c)$ |
| VIII$(n,d,c)$ | $6_0'''(h=n,d,c)$ |
| IX$(n,p,c)$ | $6''(h=n,p,c)$ |
| X$_\pm(n,d,p,c)$ | $6_\pm'''(h=n,d,p,c)$ |
| XI$_\pm(d,p,\theta,c)$ | $8''(d,p,\theta,c)$ |

**Table 3.** Coadjoint orbits and UIRs of the centrally extended planon group $G$. The table lists for every UIR of the extended planon group $G = B \ltimes A$ the coadjoint orbit from which it can be obtained by geometric quantisation. The horizontal line separates those UIRs which are essentially UIRs of Bargmann (above the line) from the rest. The orbits of type $8'''$ and 10 have $q = c$, but that was a choice. It is in fact the ratio $\lambda = q/c$ which was set to 1 via a rescaling of generators. To make more transparent contact with the UIRs we should keep the ratio arbitrary (but nonzero). The parameters $\ell$ and $c$, when they appear are assumed to be nonzero and $2\ell$ is a non-negative integer. The parameter $n$ is an integer. The two UIRs V$_\pm(d,p^\perp,E)$ and XI$_\pm(d,p,\theta,c)$ have signs which are not apparent in the corresponding coadjoint orbits. This is due to the fact that the coadjoint orbits have stabilisers with two connected components and hence they admit two inequivalent quantisations.

where $d\mu(S)$ is a bi-invariant volume form on SU(2) or, equivalently, the volume form corresponding to a round metric on $S^3$. The action of the generic element $g \in G$ is given by

$$(U(g)\psi)(E,S) = e^{i(cw+s(E-c\tilde{\varphi}))} e^{i\left(\left(\beta-\frac{1}{c}(E-c\tilde{\varphi})\boldsymbol{a}\right)\cdot S\boldsymbol{d}+\boldsymbol{a}\cdot S\boldsymbol{p}\right)} \psi(E-c\tilde{\varphi}, R^{-1}S). \tag{5.32}$$

### 5.3 Summary

It follows from the fact that the extended planon group $G = B \ltimes A$ is a regular semi-direct product (see appendix C) that all UIRs are obtained via the Mackey method and, since $A$ is abelian, Rawnsley's theorem [27] says that they can all be constructed by quantising coadjoint orbits. As usual there are a couple of caveats: not every coadjoint orbit is quantisable (e.g., there are coadjoint orbits whose angular momentum/helicity is not quantised) and a given coadjoint orbit can have inequivalent quantisations (this happens, in particular, when the stabiliser of the orbit is not connected). Short of actually quantising the coadjoint orbits, any correspondence between UIRs and coadjoint orbits remains conjectural and table 3 illustrates our attempt at such a correspondence.

JHEP11(2025)030

## 6 Discussion

In this work, we have analysed both elementary and composite systems exhibiting planon symmetry. We provided a classification of classical and quantum elementary particles by characterising the coadjoint orbits and unitary irreducible representations of the (extended) planon group. Planon symmetries are closely related to Bargmann symmetries [16], but their elementary systems are mapped into each other in quite a nontrivial way. For example the monopole, that is stuck to a point, corresponds to the massive galilean particle, which moves along straight lines, while the unrestricted dipoles correspond to massless galilean particles. In this sense, we provide a physically interesting setup for these often ignored galilean orbits.

We also present an action for composite dipole particles that naturally couples to Pretko's traceless scalar charge theory, see (4.30). Remarkably, both the free particle and its first-quantised theory exhibit a mixed Carroll-Galilei symmetry, which is fully consistent with their allowed (im)mobility (see figure 1).

Finally, we classify the planon elementary quantum systems, i.e., the unitary irreducible representations of the extended planon group and set up a correspondence between the UIRs and the coadjoint orbits.

There are various interesting avenues for further exploration:

**Other multipole symmetries and particles.** The tools we have developed here and in [21, 22, 26] can be generalised to other multipole symmetries [16] and their associated particles. A possible next step is to analyse lineons and vector charge theories [23] (see also [38–42]).

**Central extension.** In section 2.1 we showed that the planon group admits a nontrivial central extension. In the context of this work, this leads to energy that is unbounded from below. However, it is possible that alternative interpretations could give rise to interesting new physics.[8]

**Elementary versus composite.** We discussed two types of particles: elementary particles of the planon group and composite dipoles constructed from elementary monopoles. While the composite dipoles naturally couple to the traceless scalar charge gauge theory, it remains an open question whether the elementary dipoles in (3.1) also couple to a gauge theory.

A related question is whether the composite dipoles can be regarded as elementary systems of other symmetries, maybe such as those discussed in section 4.

**Soft theorems and the infrared triangle.** The scalar tensor gauge theory [22, 44] exhibits an intricate interplay between memory effects, soft theorems, and asymptotic symmetries, collectively forming an infrared triangle [45].

Following [22], one could leverage the coupling between particles and the gauge field, (4.30), along with the field theory framework (4.43), to compute memory effects and soft theorems. Notably, the study of these phenomena in $2+1$ dimensions

[8]There are even simpler systems, such as aristotelian spacetimes with one scalar charge [43], for which it would be interesting to find a physical realization.

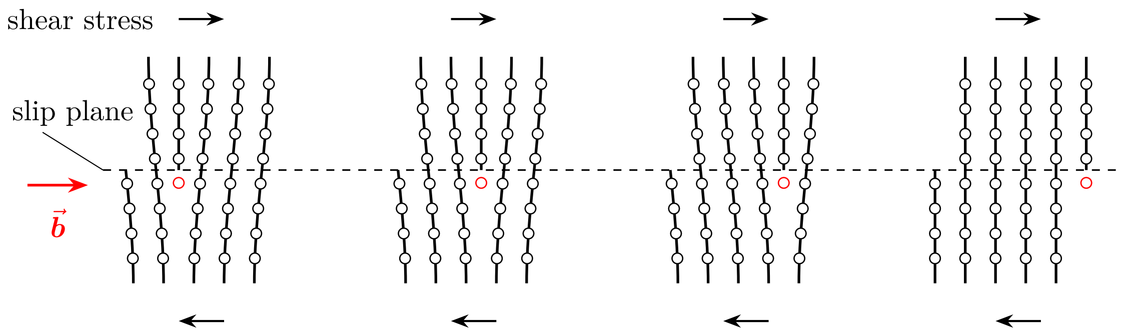


**Figure 2.** The glide motion of a dislocation (red circle) is parallel to the Burgers vector $\vec{b}$.

appears particularly promising due to potential applications in elasticity and vortex dynamics (see also [46]).

**Carroll in crystals.** Motivated by the relation between fractons and elasticity, let us provide a simple example of how Carroll and Carroll-Galilei symmetries emerge in the context of defects in crystals.

Consider a point defect in a lattice, which we can envision by just removing one atom. At zero temperature this defect is stuck to a point, which is described by the action (1.7) and which has Carroll symmetry. At higher temperatures one would need to introduce a potential that governs the now statistically allowed movement.

Let us now restrict to two spatial dimensions and consider the movement of a vacancy under a strain, as depicted in figure 2.

The movement of the vacancy is restricted to be parallel to the Burgers vector. Once we identify the Burgers vector with the dipole vector via $b_i = \epsilon_{ij} d_i$ the movement of the vacancy follows (1.8), i.e., is restricted to be orthogonal to the dipole moment. We have restricted to low temperature and (infinite) planar glide motion. For more information see, e.g., [47].

In this sense defects in crystals provide the possibly simplest physically realised model with Carroll(–Galilei) symmetry. There could be a further interesting connections between nonlorentzian geometry and condensed matter physics that await exploration.

## Acknowledgments

The work of SiP was partially supported by the Fonds de la Recherche Scientifique – FNRS under Grant No. FC.36447, the SofinaBoël Fund for Education and Talent and the *Fonds Friedmann* run by the *Fondation de l'École polytechnique*. Since 1 November 2024, SiP is supported by European Research Council (ERC) Project 101076737 — *CeleBH*.

This work was initiated while StP was supported by the Leverhulme Trust Research Project Grant (RPG-2019-218) "What is Non-Relativistic Quantum Gravity and is it Holographic?".

The research of AP is partially supported by Fondecyt grants No. 1220910 and 1230853.



JHEP11(2025)030

| Planon | Bargmann $\oplus\ \mathbb{R}$ |
|---|---|
| $L_{ab}$: spatial rotations | $L_{ab}$: spatial rotations |
| $P_a$: spatial translations | $B_a$: galilean boosts |
| $D_a$: dipole moment | $P_a$: spatial translations |
| $Z$: trace quadrupole moment | $H$: time translation |
| $Q$: electric charge | $M$: mass |
| $H$: time translation | $\mathbb{R}$ |

**Table 4.** Correspondence between planon and Bargmann algebra generators.

JMF would like to thank Jarah Fluxman for interesting conversations about the Mackey method.

AP and StP acknowledge support from the Erwin Schrödinger International Institute for Mathematics and Physics (ESI) where some of the research was undertaken during the ESI Research in Teams Programme.

## A Summary of [1] in planon language

In this appendix we summarise the results of [1] on the Bargmann group in the planon language. This is mostly an exercise in translating notation, but we think it would be useful to record the translation here.

The basic dictionary is the one relating the bases of the Lie algebras, which as mentioned already translates the Bargmann basis $\langle L_{ab}, B_a, P_a, H, M\rangle$ to $\langle L_{ab}, P_a, D_a, Z, Q\rangle$ and is summarised in table 4. The Bargmann canonical dual basis $\left\langle \lambda^{ab}, \beta^a, \pi^a, \eta, \mu \right\rangle$ translates to $\left\langle \lambda^{ab}, \pi^a, \delta^a, \zeta, \theta \right\rangle$ and hence the Bargmann moment[9]

$$\frac{1}{2} J_{ab}\lambda^{ab} + k_a\beta^a - p_a\pi^a + E\eta + m\mu \tag{A.1}$$

translates to

$$\frac{1}{2} J_{ab}\lambda^{ab} + p_a\pi^a - d_a\delta^a + \tilde{q}\zeta + q\theta. \tag{A.2}$$

The Bargmann group parameters $(a_+, a_-, \boldsymbol{a}, \boldsymbol{v}, R)$ in [1] now become $(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R)$ and the Bargmann moments $(m, E, \boldsymbol{p}, \boldsymbol{k}, J)$ in [1] now become $(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{a}, R)$.

The generic Bargmann group element in [1] is therefore now $\mathsf{b}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R)$. Its coadjoint action on the moment $\mathsf{M}(J, \tilde{q}, q, \boldsymbol{p}, \boldsymbol{d})$ is then given by

$$\mathrm{Ad}^*_{\mathsf{b}(\varphi,\tilde{\varphi},\boldsymbol{\beta},\boldsymbol{a},R)} \mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, J) = \mathsf{M}(q', \tilde{q}', \boldsymbol{d}', \boldsymbol{p}', J'), \tag{A.3}$$

[9] In [1] we chose the sign in front of $p_a\pi^a$ in order that under the boost with parameter $\boldsymbol{v}$, the momentum $\boldsymbol{p}$ would change by $\boldsymbol{p} + m\boldsymbol{v}$. We have decided to keep the minus sign in the planon case for no other reason than to simplify the translation.



JHEP11(2025)030

| # | Orbit representative $\alpha = \mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{j})$ | Stabiliser $B_\alpha$ | $\dim \mathcal{O}_\alpha$ | Equations for orbits |
|---|---|---|---|---|
| 1 | $\mathsf{M}(q_0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$ | $\{\mathbf{b}(\varphi, \tilde{\varphi}, \mathbf{0}, \mathbf{0}, R)\}$ | 6 | $q = q_0 \neq 0, \frac{1}{2q}(\\|\boldsymbol{d}\\|^2 - 2q\tilde{q}) = \tilde{q}_0, q\boldsymbol{j} = \boldsymbol{d} \times \boldsymbol{p}$ |
| 2 | $\mathsf{M}(q_0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \ell \boldsymbol{e}_3)$ | $\{\mathbf{b}(\varphi, \tilde{\varphi}, \mathbf{0}, \mathbf{0}, R) \mid R\boldsymbol{e}_3 = \boldsymbol{e}_3\}$ | 8 | $q = q_0 \neq 0, \frac{1}{2q}(\\|\boldsymbol{d}\\|^2 - 2q\tilde{q}) = \tilde{q}_0, \\|q\boldsymbol{j} - \boldsymbol{d} \times \boldsymbol{p}\\| = \ell > 0$ |
| 3 | $\mathsf{M}(0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \mathbf{0})$ | $B$ | 0 | $q = 0, \tilde{q} = \tilde{q}_0, \boldsymbol{d} = \boldsymbol{p} = \boldsymbol{j} = \mathbf{0}$ |
| 4 | $\mathsf{M}(0, \tilde{q}_0, \mathbf{0}, \mathbf{0}, \ell \boldsymbol{e}_3)$ | $\{\mathbf{b}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R) \mid R\boldsymbol{e}_3 = \boldsymbol{e}_3\}$ | 2 | $q = 0, \tilde{q} = \tilde{q}_0, \boldsymbol{d} = \boldsymbol{p} = \mathbf{0}, \\|\boldsymbol{j}\\| = \ell > 0$ |
| 5 | $\mathsf{M}(0, \tilde{q}_0, \mathbf{0}, p\boldsymbol{e}_3, h\boldsymbol{e}_3)$ | $\{\mathbf{b}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, a\boldsymbol{e}_3, R) \mid R\boldsymbol{e}_3 = \boldsymbol{e}_3\}$ | 4 | $q = 0, \tilde{q} = \tilde{q}_0, \boldsymbol{d} = \mathbf{0}, \\|\boldsymbol{p}\\| = p > 0, \boldsymbol{j} \cdot \boldsymbol{p} = hp$ |
| 6 | $\mathsf{M}(0, 0, d\boldsymbol{e}_3, \mathbf{0}, \mathbf{0})$ | $\{\mathbf{b}(\varphi, 0, \beta\boldsymbol{e}_3, \boldsymbol{a}, R) \mid R\boldsymbol{e}_3 = \boldsymbol{e}_3, \boldsymbol{a} \cdot \boldsymbol{e}_3 = 0\}$ | 6 | $q = 0, \\|\boldsymbol{d}\\| = d > 0, \boldsymbol{d} \times \boldsymbol{p} = \mathbf{0}$ |
| 7 | $\mathsf{M}(0, 0, d\boldsymbol{e}_3, p\boldsymbol{e}_2, \mathbf{0})$ | $\{\mathbf{b}(\varphi, 0, \beta\boldsymbol{e}_3, a\boldsymbol{e}_2, I)\}$ | 8 | $q = 0, \\|\boldsymbol{d}\\| = d > 0, \\|\boldsymbol{d} \times \boldsymbol{p}\\| = dp > 0$ |

**Table 5.** Coadjoint orbits of the Bargmann group $B$ in planon language. This table lists the different coadjoint orbits of the Bargmann group in planon language. In each case we exhibit an orbit representative $\alpha \in \mathfrak{b}^*$, its stabiliser subgroup $B_\alpha$ inside the Bargmann group, the dimension $\dim \mathcal{O}_\alpha$ of the orbit and the equations defining the orbit.

where

$$\begin{aligned} J' &= RJR^T + \boldsymbol{\beta}(R\boldsymbol{d} + q\boldsymbol{a})^T - (R\boldsymbol{d} + q\boldsymbol{a})\boldsymbol{\beta}^T + (R\boldsymbol{p})\boldsymbol{a}^T - \boldsymbol{a}(R\boldsymbol{p})^T \\ \boldsymbol{p}' &= R\boldsymbol{p} + q\boldsymbol{\beta} + \tilde{\varphi}(R\boldsymbol{d} + q\boldsymbol{a}) \\ \boldsymbol{d}' &= R\boldsymbol{d} + q\boldsymbol{a} \\ \tilde{q}' &= \tilde{q} + R\boldsymbol{d} \cdot \boldsymbol{a} + \frac{1}{2}q\|\boldsymbol{d}\|^2 \\ q' &= q. \end{aligned} \tag{A.4}$$

In the particular case of the spatial dimension $n = 3$, the moment $\mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{j})$ transforms into

$$\begin{aligned} \boldsymbol{j}' &= R\boldsymbol{j} - \boldsymbol{\beta} \times (R\boldsymbol{d} + q\boldsymbol{a}) + \boldsymbol{a} \times R\boldsymbol{p} \\ \boldsymbol{p}' &= R\boldsymbol{p} + q\boldsymbol{\beta} + \tilde{\varphi}(R\boldsymbol{d} + q\boldsymbol{a}) \\ \boldsymbol{d}' &= R\boldsymbol{d} + q\boldsymbol{a} \\ \tilde{q}' &= \tilde{q} + R\boldsymbol{d} \cdot \boldsymbol{a} + \frac{1}{2}q\|\boldsymbol{d}\|^2 \\ q' &= q. \end{aligned} \tag{A.5}$$

Table 1 in [1] translates into table 5.

The pull-back to the space of parameters of the Maurer-Cartan one-form on the Bargmann group (equation (4.1) in [1]) is now given by

$$b^{-1}db = \mathsf{A}(d\varphi - \boldsymbol{a}^T d\boldsymbol{\beta} - \frac{1}{2}\|\boldsymbol{a}\|^2 d\tilde{\varphi}, d\tilde{\varphi}, R^T(d\boldsymbol{\beta} + \boldsymbol{a}d\tilde{\varphi}), R^T d\boldsymbol{a}, R^T dR) \tag{A.6}$$

where $b = \mathbf{b}(\varphi, \tilde{\varphi}, \boldsymbol{\beta}, \boldsymbol{a}, R)$, and its contraction with the moment $\mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, J)$ (equation (4.2) in [1]) is now given by

$$\begin{aligned} \left\langle \mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, J), b^{-1}db \right\rangle = {} & q d\varphi - \left(\tilde{q} + \frac{1}{2}q\|\boldsymbol{a}\|^2 + (R\boldsymbol{\beta})^T \boldsymbol{a}\right) d\tilde{\varphi} \\ & - (R\boldsymbol{d} + q\boldsymbol{a})^T d\boldsymbol{\beta} + (R\boldsymbol{p})^T d\boldsymbol{a} + \frac{1}{2} \operatorname{Tr} J^T R^T dR. \end{aligned} \tag{A.7}$$



JHEP11(2025)030

In the special case of $n = 3$ dimensions, and using the isomorphism $\varepsilon : \mathbb{R}^3 \to \mathfrak{so}(3)$ defined by $\varepsilon(\boldsymbol{a})\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$, which allows us to write $J = \varepsilon(\boldsymbol{j})$, we have that

$$\begin{aligned}\left\langle \mathsf{M}(q, \tilde{q}, \boldsymbol{d}, \boldsymbol{p}, \boldsymbol{j}), b^{-1}db \right\rangle = q d\varphi - \left(\tilde{q} + \frac{1}{2} q \|\boldsymbol{a}\|^2 + R\boldsymbol{\beta} \cdot \boldsymbol{a}\right) d\tilde{\varphi} \\ - (R\boldsymbol{d} + q\boldsymbol{a}) \cdot d\boldsymbol{\beta} + (R\boldsymbol{p}) \cdot d\boldsymbol{a} + \boldsymbol{j} \cdot \varepsilon^{-1}(R^{-1}dR),\end{aligned} \tag{A.8}$$

which is simply the translation of equation (5.2) in [1].

## B Central extensions

In this appendix we collect a result necessary in section 2.3 for the classification of coadjoint orbits of the centrally extended planon group.

Let $\mathfrak{g}$ be a finite-dimensional (for definiteness) real Lie algebra and let $H^2(\mathfrak{g})$ denote the second Chevalley-Eilenberg cohomology with values in the trivial representation. This is a finite-dimensional real vector space with basis $[\omega_i]$, for $i = 1, \ldots, \dim H^2(\mathfrak{g})$, where $[\omega_i]$ is the cohomology class of the 2-cocycle $\omega_i \in \wedge^2 \mathfrak{g}^*$. This allows us to define a central extension

$$0 \longrightarrow H^2(\mathfrak{g}) \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0, \tag{B.1}$$

whose brackets are given explicitly by

$$[X, Y]_{\widehat{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} + \sum_{i=1}^{\dim H^2(\mathfrak{g})} \omega_i(X, Y) Z_i \qquad \text{and} \qquad [Z_i, -]_{\widehat{\mathfrak{g}}} = 0, \tag{B.2}$$

for all $X, Y \in \mathfrak{g}$. (We identify $\mathfrak{g}$ with a subspace of $\widehat{\mathfrak{g}}$, so in effect $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \bigoplus_i \mathbb{R}Z_i$, where the above brackets live.)

Now consider a one-dimensional central extension $\widetilde{\mathfrak{g}}$:

$$0 \longrightarrow \mathbb{R}Z \longrightarrow \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0, \tag{B.3}$$

with brackets

$$[X, Y]_{\widetilde{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} + \omega(X, Y) Z \qquad \text{and} \qquad [Z, -]_{\widetilde{\mathfrak{g}}} = 0, \tag{B.4}$$

for all $X, Y \in \mathfrak{g}$ and where $\omega \in \wedge^2 \mathfrak{g}^*$ is the corresponding 2-cocycle. In cohomology, $[\omega] = \sum_i c_i [\omega_i]$, since the $[\omega_i]$ span $H^2(\mathfrak{g})$. We can therefore choose $\omega$, perhaps by modifying it by a coboundary, so that $\omega = \sum c_i \omega_i$ as cocycles.

The claim is that there exists a Lie algebra surjective homomorphism $\varphi : \widehat{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ which is the identity on $\mathfrak{g}$; explicitly:

$$\varphi(X) = X \qquad \text{and} \qquad \varphi(Z_i) = c_i Z. \tag{B.5}$$

It follows that

$$\begin{aligned}[\varphi(X),\varphi(Y)]_{\widetilde{\mathfrak{g}}} &= [X,Y]_{\widetilde{\mathfrak{g}}} \\ &= [X,Y]_{\mathfrak{g}} + \omega(X,Y)Z \\ &= [X,Y]_{\mathfrak{g}} + \sum_i c_i\omega_i(X,Y)Z \\ &= \varphi\left([X,Y]_{\mathfrak{g}} + \sum_i \omega_i(X,Y)Z_i\right) \\ &= \varphi\left([X,Y]_{\widehat{\mathfrak{g}}}\right),\end{aligned}$$

and hence, since $Z_i$ and $Z$ are central in their respective Lie algebras, that $\varphi$ is a Lie algebra homomorphism, which is moreover manifestly surjective. The kernel of $\varphi$ consists of those $\sum_i a_i Z_i$ such that $\sum_i a_i c_i = 0$ and hence spans a hyperplane in $H^2(\mathfrak{g})$. Conversely, every hyperplane in $H^2(\mathfrak{g})$ defines a one-dimensional central extension of $\mathfrak{g}$.

## C The extended planon group is a regular semidirect product

The method of Mackey for constructing UIRs of a semidirect product $B \ltimes A$, for $A$ abelian (or more generally $A$ nilpotent) is guaranteed to produce all UIRs whenever the semidirect product is *regular*. Let $G = B \ltimes A$ be a semidirect product and let $\widehat{A}$ denote the unitary dual: the space of UIRs of $A$. Then the semidirect product $G = B \ltimes A$ is said to be **regular** if there exist a countable family $\{\mathcal{B}_i\}$ of $G$-invariant Borel subsets $\mathcal{B}_i \subset \widehat{A}$, so that every $G$-orbit $\mathcal{O} \subset \widehat{A}$ is the intersection $\bigcap_i \mathcal{B}_{n_i}$ of some subfamily of the $\{\mathcal{B}_i\}$.

In this appendix we verify that the description of the centrally extended planon group $G$ as a semidirect product $B \ltimes A$, with $B$ the Bargmann group and $A$ a two-dimensional abelian group, is regular. Since $A$ is abelian, $\widehat{A}$ can be identified with the dual of the Lie algebra $\mathfrak{a}$ of $A$, since to every $\alpha \in \mathfrak{a}^*$ we may associate the unitary one-dimensional representation with character $\chi(\exp(X)) = e^{i\alpha(X)}$, for $X \in A$.

In the case of interest, $G$ is the centrally extended planon group, $B$ is the Bargmann group and $A$ the two-dimensional abelian group with Lie algebra spanned by $H, W$. The unitary dual $\widehat{A}$ is the dual of the Lie algebra and this is a copy of the plane with coordinates $(E, c)$. As we saw in section 5, the action of the planon group on $\widehat{A}$, sends $(E, c) \mapsto (E + c\tilde{\varphi}, c)$ and hence there are two classes of orbits: point-like orbits $\mathcal{O}_{(E,0)} = \{(E, 0)\}$ and one-dimensional orbits $\mathcal{O}_{(0,c)} = \mathbb{R} \times \{c\}$ for every $c \neq 0$.

Consider the following countable family of Borel subsets of the plane:

$$\mathcal{B}_E(r,s) := \bigcup_{r<E<s} \mathcal{O}_{(E,0)} \qquad \text{and} \qquad \mathcal{B}_c(r,s) := \bigcup_{r<c<s} \mathcal{O}_{(0,c)}, \tag{C.1}$$

where $r, s \in \mathbb{Q}$ are rational numbers. The subset $\mathcal{B}_E(r, s)$ consists of those points $(E, 0)$ with $r < E < s$ and hence it is an open interval in the $c = 0$ axis. The subset $\mathcal{B}_c(r, s)$ consists of those points $(E, c)$ with $r < c < s$ and $E$ arbitrary and hence it is an open (infinite) vertical strip.

It is clear that $\mathcal{O}_{(E,0)}$ lies in every $\mathcal{B}_E(r, s)$ and hence it lies in their intersection $\bigcap_{r<E<s} \mathcal{B}_E(r, s)$. Furthermore, if $E - E' \neq 0$, there will be rational numbers $r, s$ with $r < E <$

$s$ such that $E' \notin (r,s)$ and hence $\mathcal{O}_{(E',0)} \not\subset \bigcap_{r<E<s} \mathcal{B}_E(r,s)$. Therefore, $\bigcap_{r<E<s} \mathcal{B}_E(r,s) = \mathcal{O}_{(E,0)}$.

The argument for $\mathcal{O}_{(0,c)}$ is identical. The orbit $\mathcal{O}_{(0,c)}$ belongs to every $\mathcal{B}_c(r,s)$ and hence it also belongs to the intersection $\bigcap_{r<E<s} \mathcal{B}_c(r,s)$. Moreover if $c' \neq c$, there will be some rational numbers $r,s$ with $r < c < s$ such that $c' \notin (r,s)$ and hence $\mathcal{O}_{(0,c')} \not\subset \bigcap_{r<E<s} \mathcal{B}_c(r,s)$. Therefore $\bigcap_{r<c<s} \mathcal{B}_c(r,s) = \mathcal{O}_{(0,c)}$.

In summary, every orbit lies in the intersection of a subfamily of the countable family

$$\{\mathcal{B}_E(r,s) \mid r < E < s \text{ with } r,s \in \mathbb{Q}\} \cup \{\mathcal{B}_c(r,s) \mid r < c < s \text{ with } r,s \in \mathbb{Q}\} \qquad \text{(C.2)}$$

of Borel sets of $\mathfrak{a}^*$, and hence the semidirect product $B \ltimes A$ is regular.

**Data Availability Statement.** This article has no associated data or the data will not be deposited.

**Code Availability Statement.** This article has no associated code or the code will not be deposited.



## References

[1] J.M. Figueroa-O'Farrill, S. Pekar, A. Pérez and S. Prohazka, *Galilei particles revisited*, *SciPost Phys. Lect. Notes* **93** (2025) 1 [arXiv:2401.03241] [INSPIRE].

[2] V. Bargmann, *On unitary ray representations of continuous groups*, *Annals Math.* **59** (1954) 1 [INSPIRE].

[3] C. Duval, Z. Horvath and P. Horvathy, *Geometrical spinoptics and the optical Hall effect*, *J. Geom. Phys.* **57** (2007) 925 [math-ph/0509031] [INSPIRE].

[4] R.M. Nandkishore and M. Hermele, *Fractons*, *Ann. Rev. Condens. Mat. Phys.* **10** (2019) 295 [arXiv:1803.11196] [INSPIRE].

[5] M. Pretko, X. Chen and Y. You, *Fracton Phases of Matter*, *Int. J. Mod. Phys. A* **35** (2020) 2030003 [arXiv:2001.01722] [INSPIRE].

[6] K.T. Grosvenor, C. Hoyos, F. Peña-Benitez and P. Surówka, *Space-Dependent Symmetries and Fractons*, *Front. in Phys.* **9** (2022) 792621 [arXiv:2112.00531] [INSPIRE].

[7] A. Gromov and L. Radzihovsky, *Colloquium: Fracton matter*, *Rev. Mod. Phys.* **96** (2024) 011001 [arXiv:2211.05130] [INSPIRE].

[8] M. Pretko and L. Radzihovsky, *Fracton-Elasticity Duality*, *Phys. Rev. Lett.* **120** (2018) 195301 [arXiv:1711.11044] [INSPIRE].

[9] C. Duval, G.W. Gibbons and P.A. Horvathy, *Conformal Carroll groups and BMS symmetry*, *Class. Quant. Grav.* **31** (2014) 092001 [arXiv:1402.5894] [INSPIRE].

[10] L. Donnay, A. Fiorucci, Y. Herfray and R. Ruzziconi, *Carrollian Perspective on Celestial Holography*, *Phys. Rev. Lett.* **129** (2022) 071602 [arXiv:2202.04702] [INSPIRE].



JHEP11(2025)030

[11] A. Bagchi, S. Banerjee, R. Basu and S. Dutta, *Scattering Amplitudes: Celestial and Carrollian*, *Phys. Rev. Lett.* **128** (2022) 241601 [arXiv:2202.08438] [INSPIRE].

[12] J. Figueroa-O'Farrill, E. Have, S. Prohazka and J. Salzer, *Carrollian and celestial spaces at infinity*, *JHEP* **09** (2022) 007 [arXiv:2112.03319] [INSPIRE].

[13] E. Have, K. Nguyen, S. Prohazka and J. Salzer, *Massive carrollian fields at timelike infinity*, *JHEP* **07** (2024) 054 [arXiv:2402.05190] [INSPIRE].

[14] T. Brauner et al., *Snowmass White Paper: Effective Field Theories for Condensed Matter Systems*, in the proceedings of the *Snowmass 2021*, Seattle, U.S.A., July 17–26 (2022) [arXiv:2203.10110] [INSPIRE].

[15] C. Cordova, T.T. Dumitrescu, K. Intriligator and S.-H. Shao, *Snowmass White Paper: Generalized Symmetries in Quantum Field Theory and Beyond*, in the proceedings of the *Snowmass 2021*, Seattle, U.S.A., July 17–26 (2022) [arXiv:2205.09545] [INSPIRE].

[16] A. Gromov, *Towards classification of Fracton phases: the multipole algebra*, *Phys. Rev. X* **9** (2019) 031035 [arXiv:1812.05104] [INSPIRE].

[17] L. Bidussi et al., *Fractons, dipole symmetries and curved spacetime*, *SciPost Phys.* **12** (2022) 205 [arXiv:2111.03668] [INSPIRE].

[18] A. Jain and K. Jensen, *Fractons in curved space*, *SciPost Phys.* **12** (2022) 142 [arXiv:2111.03973] [INSPIRE].

[19] C. Duval, G.W. Gibbons, P.A. Horvathy and P.M. Zhang, *Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time*, *Class. Quant. Grav.* **31** (2014) 085016 [arXiv:1402.0657] [INSPIRE].

[20] E. Bergshoeff, J. Gomis and G. Longhi, *Dynamics of Carroll Particles*, *Class. Quant. Grav.* **31** (2014) 205009 [arXiv:1405.2264] [INSPIRE].

[21] J. Figueroa-O'Farrill, A. Pérez and S. Prohazka, *Carroll/fracton particles and their correspondence*, *JHEP* **06** (2023) 207 [arXiv:2305.06730] [INSPIRE].

[22] A. Pérez, S. Prohazka and A. Seraj, *Fracton Infrared Triangle*, *Phys. Rev. Lett.* **133** (2024) 021603 [arXiv:2310.16683] [INSPIRE].

[23] M. Pretko, *Generalized Electromagnetism of Subdimensional Particles: a Spin Liquid Story*, *Phys. Rev. B* **96** (2017) 035119 [arXiv:1606.08857] [INSPIRE].

[24] A. Kumar and A.C. Potter, *Symmetry-enforced fractonicity and two-dimensional quantum crystal melting*, *Phys. Rev. B* **100** (2019) 045119 [arXiv:1808.05621] [INSPIRE].

[25] M. Pretko, Z. Zhai and L. Radzihovsky, *Crystal-to-Fracton Tensor Gauge Theory Dualities*, *Phys. Rev. B* **100** (2019) 134113 [arXiv:1907.12577] [INSPIRE].

[26] J. Figueroa-O'Farrill, A. Pérez and S. Prohazka, *Quantum Carroll/fracton particles*, *JHEP* **10** (2023) 041 [arXiv:2307.05674] [INSPIRE].

[27] J.H. Rawnsley, *Representations of a semi-direct product by quantization*, *Math. Proc. Cambridge Phil. Soc.* **78** (1975) 345.

[28] J.-M. Souriau, *Structure des systèmes dynamiques*, Maîtrises de mathématiques, Dunod, Paris (1970).

[29] K. Morand, *Possible ambient kinematics*, *J. Math. Phys.* **64** (2023) 112502 [arXiv:2305.14371] [INSPIRE].



JHEP11(2025)030

[30] B. Oblak, *BMS Particles in Three Dimensions*, Ph.D. thesis, Brussels University, Belgium (2016) [arXiv:1610.08526] [INSPIRE].

[31] G. Barnich, K. Nguyen and R. Ruzziconi, *Geometric action for extended Bondi-Metzner-Sachs group in four dimensions*, JHEP **12** (2022) 154 [arXiv:2211.07592] [INSPIRE].

[32] T. Basile, E. Joung and T.H. Oh, *Manifestly covariant worldline actions from coadjoint orbits. Part I. Generalities and vectorial descriptions*, JHEP **01** (2024) 018 [arXiv:2307.13644] [INSPIRE].

[33] I.A. Lahlali and J.A. O'Connor, *Coset symmetries and coadjoint orbits*, arXiv:2411.05918 [INSPIRE].

[34] G. Barnich and T. Smoes, *Model spaces as constrained Hamiltonian systems. Part I. Application to SU(2)*, JHEP **10** (2025) 056 [arXiv:2502.20267] [INSPIRE].

[35] J. de Boer et al., *Carroll Symmetry, Dark Energy and Inflation*, Front. in Phys. **10** (2022) 810405 [arXiv:2110.02319] [INSPIRE].

[36] D.H. Whittum, *Introduction to electrodynamics for microwave linear accelerators*, in the proceedings of the *Joint CERN-US-Japan Accelerator School: Course on Frontiers of Accelerator Technology: RF Engineering for Particle Accelerators*, Tsukuba, Japan, September 09–18 (1996) [INSPIRE].

[37] A.O. Barut and R. Rączka, *Theory of group representations and applications*, PWN — Polish Scientific Publishers, Warsaw (1977) [INSPIRE].

[38] E. Bertolini and N. Maggiore, *Maxwell theory of fractons*, Phys. Rev. D **106** (2022) 125008 [arXiv:2209.01485] [INSPIRE].

[39] S.A. Baig et al., *Spacetime Subsystem Symmetries*, arXiv:2303.15590 [INSPIRE].

[40] O. Kasikci, M. Ozkan and Y. Pang, *Carrollian origin of spacetime subsystem symmetry*, Phys. Rev. D **108** (2023) 045020 [arXiv:2304.11331] [INSPIRE].

[41] M.M. Ahmadi-Jahmani and A. Parvizi, *Fracton and non-Lorentzian particle duality: gauge field couplings and geometric implications*, JHEP **08** (2025) 157 [arXiv:2503.21660] [INSPIRE].

[42] E.A. Bergshoeff, G. Giorgi, J. Rosseel and P. Salgado-Rebolledo, *A conformal approach to matter coupled Aristotelian gravity*, arXiv:2507.16943 [INSPIRE].

[43] J. Figueroa-O'Farrill, R. Grassie and S. Prohazka, *Lifshitz symmetry: Lie algebras, spacetimes and particles*, SciPost Phys. **14** (2023) 035 [arXiv:2206.11806] [INSPIRE].

[44] A. Pérez and S. Prohazka, *Asymptotic symmetries and soft charges of fractons*, Phys. Rev. D **106** (2022) 044017 [arXiv:2203.02817] [INSPIRE].

[45] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, Princeton University Press (2018) [INSPIRE].

[46] L. Tsaloukidis and P. Surówka, *Elastic Liénard-Wiechert potentials of dynamical dislocations from tensor gauge theory in 2+1 dimensions*, Phys. Rev. B **109** (2024) 104118 [arXiv:2302.14092] [INSPIRE].

[47] D. Hull and D. Bacon, *Introduction to Dislocations*, Butterworth-Heinemann (2011).