Fracton Infrared Triangle

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In theories with conserved dipole moment, isolated charged particles (fractons) are immobile, but dipoles can move. We couple these dipoles to the fracton gauge theory and analyze the universal infrared structure. This uncovers an observable double kick memory effect which we relate to a novel dipole soft theorem. Together with their asymptotic symmetries this constitutes the first realization of an infrared triangle beyond Lorentz symmetry. This demonstrates the robustness of these IR structures and paves the way for their investigation in condensed matter systems and beyond.

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Introduction.—Fractons [1,2] are novel quasiparticles whose characteristic feature is their limited mobility. This restricted mobility originates from their built-in *dipole symmetry* which leads to conserved dipole moments $d^i = \int x^i \rho d^3x$. Fracton theories attract attention not only for their interesting phenomenological applications, but also for their intricate theoretical underpinnings which challenge common quantum field techniques [3–5]. The dipole symmetry is an example of generalized symmetries, whose study has led to breakthroughs in our understanding of quantum field theories [6,7]. Moreover, the underlying symmetries are closely related to Carroll symmetries [8–11], which play a fundamental role in flat-space holography [12–15].

The infrared (IR) triangle [16] is a triangular correspondence, which connects asymptotic symmetries, memory effects, and soft theorems. It controls the infrared behavior of gravity and several relativistic gauge theories, and is a building block of the celestial holography program [17–21].

In [22] asymptotic symmetries of the fracton gauge theory were analyzed at spatial infinity and the existence of soft charges already hinted at an infrared triangle, which we establish in the current work, by uncovering a novel and observable double kick memory effect, which we relate to a dipole soft theorem and to novel asymptotic symmetries in the radiation regimes. This shows that IR triangles indeed persist beyond the Lorentzian setup and lead to exciting new physics.

One consequence of the non-Lorentzian nature of the fracton gauge theory is that its degrees of freedom have two different dispersion relations and propagation speeds. As a result, the spacetime has two radiative regions (see Fig. 1), in contradistinction to the single null infinity in Lorentzian theories. This makes the asymptotic structure of these theories richer than that of their relativistic counterparts.

One of our main results is a novel memory effect. Memory effects refer to observables that persist in a probe system after the passage of waves. Early examples of memory effects in the context of gravity include the displacement of freely falling test masses [23–26]. However, this field has witnessed a significant interest in recent years due the to discovery of new memory effects and their relation to fundamental properties of gravity [16,27–40]. While fracton memory effect shares some features with gravity and some with gauge theories, it has unique properties, in particular, it leads to a *double kick* effect on test quasiparticles (dipoles),

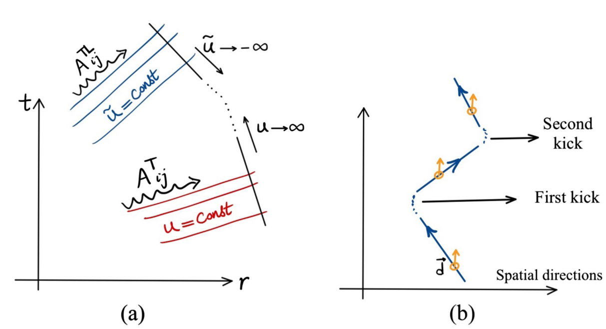

FIG. 1. (a) Since the waves propagate with speeds c and $\tilde{c}$ the theory has two different radiation zones (two "null infinities"). (b) It follows that a dipole in the far region will receive two kicks, but the orientation $\vec{d}$ of the dipole stays inert.

as is depicted in Fig. 1. This double kick memory effect is a measurable infrared observable and encourages the exciting perspective to try to observe them in condensed matter systems, especially considering the ongoing experimental investigations into systems with dipole symmetry [41–44].

Fracton gauge theory.—In this section, we introduce the (scalar charge) gauge theory [45,46] which describes the interactions among charged fractons. It is a higher-rank gauge theory defined by a scalar field ϕ and a symmetric tensor A_{ij} (i, j, k are spatial indices from 1 to 3), with the Lagrangian density

$$\mathcal{L}[A_{ij},\phi]=\frac{1}{2}E_{ij}E^{ij}-\frac{c^2}{4}F_{ijk}F^{ijk}+\phi\rho-A_{ij}J^{ij}, \quad (1)$$

where $E_{ij}=\partial_t A_{ij}-\partial_i\partial_j\phi$ and $F_{ijk}=2\partial_{[i}A_{j]k}$. The constant c has units of velocity and the tensors E_{ij} and F_{ijk} are analogous to electric and magnetic tensors, respectively. The symmetries imply $F_{[ijk]}=0$ and the useful relation $F_{i[jk]}=-\frac{1}{2}F_{jki}$. The action (1) is invariant under the gauge transformation

$$\delta_\Lambda\phi=\partial_t\Lambda \quad \delta_\Lambda A_{ij}=\partial_i\partial_j\Lambda, \quad (2)$$

and leads to the equations of motion

$$\partial_i\partial_j E^{ij}=\rho, \quad (3a)$$

$$\partial_t E_{ij}-c^2\partial_k F^k_{(ij)}=-J_{ij}, \quad (3b)$$

where ρ and J_{ij} represent the charge and current densities, respectively. Consistency of the field equations (3) leads to the continuity equation $\partial_t\rho+\partial_i\partial_j J^{ij}=0$ which implies that the electric charge Q and dipole moment d^i

$$Q=\int\rho d^3x, \quad d^i=\int x^i\rho d^3x \quad (4)$$

of a localized source are conserved. The conservation of the dipole moment implies, in particular, that isolated monopoles in this theory cannot move. Existence and time independence of monopole and dipole charges (4) are guaranteed by imposing asymptotic falloff conditions $\rho=O(1/r^{4+\epsilon})$ and $J_{ij}=O(1/r^{2+\epsilon})$ for $\epsilon>0$.

Decoupled field equations and memory effect.—An important consequence of the fact that the fracton gauge theory is not Lorentz invariant is that various dynamical degrees of freedom obey different dynamical equations. We decouple the equations of motion using a systematic decomposition of the gauge field into representations of the rotation group as $A_{ij}=A^{\mathrm{T}}_{ij}+A^{\mathrm{TL}}_{ij}+A^{\mathrm{L}}_{ij}$, where the superscripts T, TL and L, denoted collectively by $\blacksquare$, refer to transverse, transverse-longitudinal, and longitudinal projections $A^{\blacksquare}_{ij}=P^{\blacksquare}_{ijmn}A_{mn}$ with

$$P^{\mathrm{T}}_{ijmn}=P_{m(i}P_{j)n}, \quad P^{\mathrm{TL}}_{ijmn}=2P_{m(i}\Pi_{j)n}, \quad P^{\mathrm{L}}_{ijmn}=\Pi_{m(i}\Pi_{j)n}, \quad (5)$$

defined in terms of longitudinal and transverse projectors $\Pi_{ij}=\partial_i\Delta^{-1}\partial_j$, $P_{ij}=\delta_{ij}-\Pi_{ij}$. While the T, TL projections are gauge invariant, the longitudinal component can be written as $A^{\mathrm{L}}_{ij}=\partial_i\partial_j\psi$, which is shifted under gauge transformations $\delta_\Lambda\psi=\Lambda$. Using this decomposition, the constraint equation (3a) reduces to $\Delta\Delta(\dot\psi-\phi)=\rho$, which implies that the gauge-invariant combination $\dot\psi-\phi$ is nondynamical and the electric field is hence given by

$$E_{ij}=\dot A^{\mathrm{T}}_{ij}+\dot A^{\mathrm{TL}}_{ij}+\partial_i\partial_j\Delta^{-1}(\Delta^{-1}\rho). \quad (6)$$

Overdots denote time derivatives and Δ^{-1} is the inverse of the Laplacian $\Delta=\partial_i\partial^i$, given by a Green function integral. Using (6) in (3b) combined with $\dot F_{ijk}=2\partial_{[i}E_{j]k}$, one finds the decoupled dynamical equations

$$\Box_c A^{\mathrm{T}}_{ij}=J^{\mathrm{T}}_{ij}, \quad \Box_{\tilde c}A^{\mathrm{TL}}_{ij}=J^{\mathrm{TL}}_{ij}, \quad (7)$$

where $\Box_c\equiv-\partial_t^2+c^2\Delta$ is the wave operator with speed c. The unequal propagation speeds c and $\tilde c=c/\sqrt{2}$ of the dynamical degrees of freedom reflect the non-Lorentzian nature of this theory with Aristotelian symmetry structure [8,47]. The dynamical variables A^{T}_{ij}, A^{TL}_{ij} are gauge invariant and account for the expected $2+1$ and 2 degrees of freedom, respectively. Equations (7) are solved by

$$A^{\mathrm{T}}_{ij}=P^{\mathrm{T}}_{ijmn}\Box_c^{-1}J_{mn}, \quad A^{\mathrm{TL}}_{ij}=P^{\mathrm{TL}}_{ijmn}\Box_{\tilde c}^{-1}J_{mn}, \quad (8)$$

where $\Box_c^{-1}$ represents the inverse of $\Box_c$ using a retarded Green function integral. In deriving (6) and (8), we have used the commutativity of derivatives with Δ^{-1}, $\Box_c^{-1}$, which can be proven in Fourier space.

Asymptotic behavior.—Assuming that the source is localized, the asymptotic behavior of the fields can be derived from an asymptotic expansion of (8) as $r\to\infty$. The Coulombic contribution to (6) takes the form

$$\Delta^{-1}(\Delta^{-1}\rho)=-\int\frac{d^3x'}{8\pi}|\boldsymbol{x}-\boldsymbol{x}'|\rho(\boldsymbol{x}')=-\frac{Qr}{8\pi}+O(r^0) \quad (9)$$

and therefore

$$E_{ij}=\frac{1}{r}\left[\dot{\bar A}^{\mathrm{T}}_{ij}+\dot{\bar A}^{\mathrm{TL}}_{ij}-\frac{Q}{8\pi}\bar P_{ij}\right]+O(1/r^2), \quad (10)$$

where $\bar A^{\blacksquare}_{ij}$ refers to the leading order behavior $A^{\blacksquare}_{ij}=(1/r)\bar A^{\blacksquare}_{ij}+O(1/r^2)$. For the electric field, the nonlocal projectors $P^{\mathrm{T/TL}}_{ijmn}$ reduce, at leading order, to local projections on the sphere taking the same form as (5) but with

(Π_{ij}, P_{ij}) replaced by $(\bar{\Pi}_{ij} = n_i n_j, \bar{P}_{ij} = \delta_{ij} - \bar{\Pi}_{ij})$ with $n^i = (x^i/r)$ the normal radial vector.

Radiation of scattering dipoles: Since in this theory isolated monopoles cannot move, a natural setup is the scattering of moving dipoles. The current of a dipole d^i moving on a path $z^i(t)$ with velocity $v^i = \dot{z}^i(t)$ is given by [46]

$$\rho(t, \boldsymbol{x}) = -d^i \partial_i \delta^3(\boldsymbol{x} - \boldsymbol{z}(t)) \tag{11a}$$

$$J_{ij}(t, \boldsymbol{x}) = -d_{(i} v_{j)} \delta^3(\boldsymbol{x} - \boldsymbol{z}(t)). \tag{11b}$$

We can use the current to calculate the Green integral

$$\Box_c^{-1} J_{ij} = \frac{1}{4\pi c^2} \frac{d_{(i} v_{j)}}{R(1 - \hat{\boldsymbol{R}} \cdot \boldsymbol{v}/c)} \bigg|_{t_{\text{ret}} = t - R/c}, \tag{12}$$

where $\boldsymbol{R} \equiv \boldsymbol{x} - \boldsymbol{z}(t)$ and $R, \hat{\boldsymbol{R}}$ are its norm and unit direction, respectively. This is the fracton dipole analog of the Liénard-Wiechert solution of moving point charges in electrodynamics. Inserting (12) into (8), one finds that far from the source, asymptotic fields $\bar{A}^{\text{T}}_{ij}(\boldsymbol{n})$ and $\bar{A}^{\text{TL}}_{ij}(\boldsymbol{n}) = n_{(i} \bar{A}^{\text{TL}}_{j)}$ of a dipole moving with constant velocity are given by

$$\bar{A}^{\text{T}}_{ij} = \frac{1}{4\pi c^2} \frac{d^{\perp}_{(i} v^{\perp}_{j)}}{1 - \boldsymbol{n} \cdot \boldsymbol{v}/c}, \quad \bar{A}^{\text{TL}}_i = \frac{1}{4\pi \tilde{c}^2} \frac{d^{\perp}_i v_r + v^{\perp}_i d_r}{1 - \boldsymbol{n} \cdot \boldsymbol{v}/\tilde{c}}, \tag{13}$$

where we use $X^{\perp}_i \equiv \bar{P}_{ij} X^j$ and $X_r \equiv X_i n^i$.

Memory effects: The scattering process of N dipoles labeled by $\alpha = \{1, \ldots, N\}$ will induce a memory $\delta \bar{A}^{\text{T}}_{ij} = \lim_{u \to \infty} [\bar{A}^{\text{T}}_{ij}(u) - \bar{A}^{\text{T}}_{ij}(-u)]$, $\delta \bar{A}^{\text{TL}}_{ij} = \lim_{\tilde{u} \to \infty} [\bar{A}^{\text{TL}}_{ij}(\tilde{u}) - \bar{A}^{\text{TL}}_{ij}(-\tilde{u})]$ given by

$$\delta \bar{A}^{\text{T}}_{ij}(\boldsymbol{n}) = \frac{1}{4\pi c^2} \sum_{\alpha=1}^{N} \frac{\eta^\alpha d^{\alpha\perp}_{(i} v^{\alpha\perp}_{j)}}{1 - \boldsymbol{n} \cdot \boldsymbol{v}^\alpha/c}, \tag{14a}$$

$$\delta \bar{A}^{\text{TL}}_i(\boldsymbol{n}) = \frac{1}{4\pi \tilde{c}^2} \sum_{\alpha=1}^{N} \frac{\eta^\alpha (d^{\alpha\perp}_i v^\alpha_r + v^{\alpha\perp}_i d^\alpha_r)}{1 - \boldsymbol{n} \cdot \boldsymbol{v}^\alpha/\tilde{c}}, \tag{14b}$$

where $\eta = 1$ for outgoing and -1 for incoming dipoles and $u \coloneqq t - r/c$, $\tilde{u} \coloneqq t - r/\tilde{c}$.

Double kick memory effect: These memory effects have observable consequences. As an example, consider a fractonic particle with dipole moment d^i at a large distance r from the source of radiation, which has initial velocity v^i_0 at some initial time $t = t_0$. The dipole is affected by the radiation through the generalized Lorentz force law [46]

$$\dot{v}_i = -\frac{d^j}{m} (E_{ij} + v^k F_{kij}). \tag{15}$$

Using the asymptotic form (10) of the electric field and an analogous expression for the magnetic field, and integrating the result over time in the interval (t_0, t_f), we find that there is a net kick effect on the dipole that is proportional to the memory effects [up to order $O(r^{-2})$]

$$\delta v_r = -\frac{d^i}{mr} \left(\delta \bar{A}^{\text{TL}}_i - \frac{v^j_0}{c} \delta \bar{A}^{\text{T}}_{ij} \right), \tag{16a}$$

$$\delta v^{\perp}_i = \frac{-1}{mr} \left[d_r \delta \bar{A}^{\text{TL}}_i + d^j \delta \bar{A}^{\text{T}}_{ij} \left(1 + \frac{v^r_0}{c} \right) - \frac{q d^{\perp}_i}{8\pi} \delta t \right], \tag{16b}$$

where $\delta t = t_f - t_0$. Each of the fast and slow radiative modes cause a net kick effect on the test dipole and thus it undergoes a *double kick* memory effect (see Fig. 1).

Asymptotic conditions and Bondi analysis of fractonic waves.—An immediate consequence of the different propagation speeds in this theory is that at very large distances, there will be a decoupling of the T and TL sectors, defining two distinct radiation zones: the fast radiation zone of T waves, where $t, r \to \infty$ while $u \coloneqq t - r/c$ remains finite, and the slow radiation zone of TL waves, with $\tilde{u} \coloneqq t - r/\tilde{c}$ finite. Accordingly, the radiative phase space splits into two distinct phase spaces $\Gamma_{\text{T}}, \Gamma_{\text{TL}}$.

We will therefore analyze the structure of fields and asymptotic symmetries in each of these asymptotic regions independently. To this end, we solve (7) asymptotically in the limit $r \to \infty$. We assume that the source fields decay fast enough, so that we can implement source-free wave equations at leading orders. We use the notation $(c_\blacksquare, u_\blacksquare)$ to unify results that are valid in both radiative regions with their respective propagation speed and retarded time.

The transformation from Cartesian to spherical coordinates, which are more convenient for the asymptotic analysis is carried out by suitable projections with the triad $(n^i, r e_A{}^i)$ and $e^i_A(\boldsymbol{n}) = (\partial n^i / \partial \theta^A)$. The induced metric $\gamma_{AB} = \delta_{ij} e^i_A e^j_B$ denotes the metric of the unit 2-sphere $\gamma_{AB} dx^A dx^B = d\theta^2 + \sin^2\theta d\varphi^2$ which is used to lower and raise $A, B, \ldots$ indices and has determinant γ. Therefore, the analysis in the preceding sections reveals the following asymptotic behavior for the electric field written as a tensor density, i.e., multiplied by $r^2 \sqrt{\gamma}$

$$E^{rr} = \bar{E}^{rr} + r^{-1} E^{rr}_{(-1)} + O(r^{-2}), \tag{17a}$$

$$E^{rA} = \bar{E}^{rA} + r^{-1} E^{rA}_{(-1)} + O(r^{-2}), \tag{17b}$$

$$E^{AB} = r^{-1} \bar{E}^{AB} + r^{-2} E^{AB}_{(-2)} + O(r^{-3}). \tag{17c}$$

The asymptotic behavior of the electric field is consistent with the following falloff for the potentials

$$\phi = \frac{q}{8\pi} r + \phi^{(0)} + O(r^{-1}), \quad A_{rr} = O(r^{-2}), \tag{18a}$$

$$A_{rA} = \bar{A}_{rA} + O(r^{-1}), \quad A_{AB} = r \bar{A}_{AB} + O(r^0). \tag{18b}$$

Viewing $A_{rr} = O(r^{-2})$ as a gauge-fixing condition, (18) is consistent with the asymptotic behaviors of (8) and (9), and the parameter q is matched with the total charge Q of the bulk solution. As shown in [22], this expression for the leading order of ϕ is essential for achieving both finite energy density and charges.

The falloff (18) is preserved under gauge transformations (2) with parameter of the form

$$\Lambda = r\lambda(\theta,\varphi) + uc\eta(\theta,\varphi) + \epsilon(\theta,\varphi) + O(r^{-1}), \tag{19}$$

and a similar expression in the slow radiation zone by replacing (u, c) by $(\tilde{u}, \tilde{c})$ and (λ, η) by $(\tilde{\lambda}, \tilde{\eta})$. The corresponding gauge transformation take the same functional form in both regions [except that $(\lambda, \eta) \to (\tilde{\lambda}, \tilde{\eta})$ in the TL sector]

$$\delta_\Lambda \bar{A}_{rA} = -D_A \eta, \tag{20a}$$

$$\delta_\Lambda \bar{A}_{AB} = D_{AB}\lambda + \gamma_{AB}\left(\frac{1}{2}(D^2+2)\lambda - \eta\right), \tag{20b}$$

where D_A is the covariant derivative with respect to γ_{AB}, while $D^2 = D_A D^A$ is the sphere Laplacian and $D_{AB} \equiv \frac{1}{2}(D_A D_B + D_B D_A - \gamma_{AB} D^2)$. The corresponding charges and fluxes should be worked out in each radiative region separately, as the radiative fields behave differently:

Transverse sector T: The radiative field in this region is $\bar{A}_{AB}(u, \theta^A)$, while $\bar{A}_{rA}$ and $\phi^{(0)}$ are time independent functions on the sphere, we therefore find

$$\bar{E}^{rA} = 0 \quad \bar{E}^{AB} = -\sqrt{\gamma}\gamma^{AB}\frac{q}{8\pi} + \sqrt{\gamma}\dot{\bar{A}}^{AB}. \tag{21a}$$

Transverse-longitudinal sector TL: The radiative field in this region is $\bar{A}_{rA}(\tilde{u}, \theta^A)$, while $\bar{A}_{AB}$ and $\phi^{(0)}$ are time independent. Accordingly,

$$\bar{E}^{rA} = \sqrt{\gamma}\gamma^{AB}\dot{\bar{A}}_{rB}, \quad \bar{E}^{AB} = -\sqrt{\gamma}\gamma^{AB}\frac{q}{8\pi}. \tag{22}$$

The charges corresponding to asymptotic symmetries (19) can be worked out using canonical or covariant phase space methods [48,49]. To get finite charges, we also impose the following constraints [22], consistent with the solutions obtained in the previous section

$$\bar{A}, \dot{A}^{(0)}, \phi^{(0)}, \lambda, \eta \,\text{contain}\, \ell \geq 1 \,\text{harmonics} \tag{23}$$

in a harmonic expansion in $Y_{\ell,m}(\theta,\phi)$. Using the equation of motion

$$\frac{1}{c_\blacksquare}\dot{\bar{E}}^{rr} + \bar{E} - 2D_A\bar{E}^{rA} = -\sqrt{\gamma}\frac{q}{4\pi}, \tag{24}$$

together with the conditions (21) or (22), one finds that the total charge is finite and reads

$$Q[\epsilon,\lambda,\eta] = \oint d^2x\left(-\epsilon\sqrt{\gamma}\frac{q}{4\pi} + \lambda\mathcal{P} + \eta\mathcal{Q}\right), \tag{25}$$

where the charge densities depending on the radiation zone are given by

$$\mathcal{P} := \bar{E}^{rr} - 2D_A E^{rA}_{(-1)} + E_{(-2)} + \frac{1}{c_\blacksquare}\dot{E}^{rr}_{(-1)}, \qquad \mathcal{Q} := -\bar{E}^{rr}.$$

The term proportional to ϵ in (25) gives the total charge, while terms proportional to λ and η give two infinite sets of charges in each sector, where the conserved dipole moment is the $\ell = 1$ in the mode expansion of $\mathcal{P}$.

In the presence of radiation, charges are no longer conserved, but carried away by fractonic waves. The time evolution of the charges are specified by the following flux equations derived from the equations of motion. In the fast radiation zone (T sector)

$$\dot{\mathcal{P}} = -\sqrt{\gamma}(D^A D^B + \gamma^{AB})\dot{\bar{A}}_{AB}, \quad \dot{\mathcal{Q}} = c\sqrt{\gamma}\dot{\bar{A}}, \tag{26}$$

whereas in the slow radiation zone (TL sector)

$$\dot{\mathcal{P}} = 0, \quad \dot{\mathcal{Q}} = -2\tilde{c}\sqrt{\gamma}D^A\dot{\bar{A}}_{rA}. \tag{27}$$

Note that (26) ensures the conservation of the dipole moment (the $\ell = 1$ modes of $\mathcal{P}$). Specifically, the TT component of $\dot{\bar{A}}_{AB}$ does not contain modes with $\ell = 1$, while the $\ell = 1$ mode of its trace is annihilated by the operator $D^2 + 2$ in front of it.

According to the flux-balance equations

$$\frac{dE}{dt} = -c\oint d^2x\sqrt{\gamma}\dot{\bar{A}}^{AB}\dot{\bar{A}}_{AB}, \quad (\text{T sector}), \tag{28a}$$

$$\frac{dE}{dt} = -2\tilde{c}\oint d^2x\sqrt{\gamma}\gamma^{AB}\dot{\bar{A}}_{rA}\dot{\bar{A}}_{rB}, \quad (\text{TL sector}), \tag{28b}$$

implying that radiation carries away energy from the system, which is the fractonic analogue of Bondi's energy loss formula [50,51]. Note that the fluxes have a definite sign, as expected.

Memory effect and asymptotic symmetries.—In this section, we show that fracton memory effects can be realized as a vacuum transition under fracton asymptotic symmetries. Consider a dynamical process in which the system is nonradiative before some initial time and after some final time, implying that $\dot{\bar{A}}_{AB} = 0$ in the limit $u \to \pm\infty$ and $\dot{\bar{A}}_{rA} = 0$ in the limit $\tilde{u} \to \pm\infty$. The memory effect discussed in the section on radiation of scattering dipoles implies that the vacua are not identical before and after radiation. Rather, the transition between the vacua induces a large gauge transformation as we will see shortly.

Starting from (14) and transforming to spherical coordinates, we find that the memory terms can be expressed in terms of three scalar *memory fields* $C_{\mathrm{S}}, C_{\mathrm{V}}, C_{\mathrm{T}}$ on the sphere (S, V, T refer to scalar, vector, and tensor modes)

$$\begin{aligned} &\delta\bar{A}_{AB} = D_{AB}C_{\mathrm{T}} + \frac{1}{2}\gamma_{AB}C_{\mathrm{S}} \quad \delta\bar{A}_{rA} = 0, \quad (\mathrm{T\,sector}), \\ &\delta\bar{A}_{rA} = D_{A}C_{\mathrm{V}} \quad \delta\bar{A}_{AB} = 0 \quad (\mathrm{TL\,sector}). \end{aligned} \tag{29}$$

The first (second) line refers to the memory accumulated during the fast (slow) radiation.

Memory as vacuum transition: In each sector, the corresponding memory leads to a change of vacuum given by a large gauge symmetry. The memory $\delta\bar{A}_{rA}$ in the TL sector corresponds to a large gauge transformation given by $\tilde{\eta} = -C_{\mathrm{V}}, \tilde{\lambda} = 0$ in (20). The T sector is subtle, the memory $\delta\bar{A}_{AB}$ corresponds to choosing $\lambda = C_{\mathrm{T}}$, $\eta = \frac{1}{2}[(D^2 + 2)C_{\mathrm{T}} - C_{\mathrm{S}}]$ in (20). This choice also induces a change in $\bar{A}_{rA}$, but that is no problem since $\bar{A}_{rA}$ is not part of the radiative phase space of the T sector. These equations can be inverted to compute the memory fields

$$C_{\mathrm{V}}(\boldsymbol{n}) = \frac{n^i}{4\pi}\int d\Omega' \frac{\delta\bar{A}^{\mathrm{TL}}_i(\boldsymbol{n}')}{1 - \boldsymbol{n}\cdot\boldsymbol{n}'}, \tag{30a}$$

$$C_{\mathrm{T}}(\boldsymbol{n}) = \frac{n^i n^j}{4\pi}\int d\Omega' \frac{\delta\bar{A}^{\mathrm{TT}}_{ij}(\boldsymbol{n}')}{1 - \boldsymbol{n}\cdot\boldsymbol{n}'}, \tag{30b}$$

$$C_{\mathrm{S}}(\boldsymbol{n}) = \bar{P}^{ij}\delta\bar{A}^{\mathrm{T}}_{ij}(\boldsymbol{n}), \tag{30c}$$

where $A \equiv \bar{P}^{ij}\bar{A}^{T}_{ij}$ and $\bar{A}^{\mathrm{TT}}_{ij} = \bar{A}^{\mathrm{T}}_{ij} - \frac{1}{2}\bar{P}_{ij}A$. Implementing (14) in these equations reveals the memory fields. The change in the charges resulting from radiation flux is also exclusively determined by the memory fields. Integrating (26) and (27) in time, and using (29), one finds that the flux of charges through the fast radiation (T sector) is given by

$$\delta\mathcal{P} = -\frac{1}{2}c\sqrt{\gamma}(D^2 + 2)(D^2 C_{\mathrm{T}} + 2C_{\mathrm{S}}), \tag{31a}$$

$$\delta\mathcal{Q} = 2c\sqrt{\gamma}C_{\mathrm{S}}, \tag{31b}$$

while the slow radiation (TL sector) carries

$$\delta\mathcal{Q} = -2\tilde{c}\sqrt{\gamma}D^2 C_{\mathrm{V}}, \quad \delta\mathcal{P} = 0. \tag{32}$$

Thus we have established connections between asymptotic symmetries and memory effects in the fracton infrared triangle.

Another important aspect to consider is the matching conditions at the intersection of the different radiation regions. We expect to discuss this point in detail in the extended version of our work [52].

Soft factors from the memory effect.—In this section, we determine the soft factors for a scattering process of dipoles from the memory effect.

Consider a scattering process of N dipoles with momentum $\boldsymbol{p}_\alpha = m_\alpha \boldsymbol{v}_\alpha$ and dipole moment $\boldsymbol{d}_\alpha$, with the emission of one fractonic soft photon with frequency ω and polarization projectors $\epsilon^{ij}_{\blacksquare}$. The scattering amplitude is expected to factorize in the soft limit $\omega \to 0$ as

$$\mathcal{A}_{n+1}(\boldsymbol{v}_\alpha, \boldsymbol{d}_\alpha; \omega, \epsilon^{ij}_{\blacksquare}) = \frac{1}{\omega}\epsilon^{ij}_{\blacksquare}S^{\blacksquare}_{ij}\mathcal{A}_n(\boldsymbol{v}_\alpha, \boldsymbol{d}_\alpha) + O(\omega^0). \tag{33}$$

To derive the soft factor $S^{\blacksquare}_{ij}$, we will closely follow [30]. We illustrate the derivation for the T sector, while the analysis for the TL sector would be similar. Starting from a Fourier mode decomposition of the gauge field A^{T}_{ij}, it can be shown that the radiative field $\bar{A}^{\mathrm{T}}_{ij} = \lim_{r\to\infty}(r\bar{A}^{\mathrm{T}}_{ij})$ can be computed using a saddle point approximation

$$\bar{A}^{\mathrm{T}}_{ij}(u, \boldsymbol{n}) = \frac{i}{2}\int \frac{d\omega}{(2\pi c)^2}\left(\epsilon^{\alpha}_{ij}a^{\dagger}_{\alpha}(\omega, \boldsymbol{n})e^{i\omega u} - \mathrm{c.c.}\right),$$

where α labels the transverse mode with polarization ϵ^{α}_{ij}, created by $a^{\dagger}_{\alpha}$. The next step is to use this result to compute the memory

$$\delta\bar{A}^{\mathrm{T}}_{ij} = \int du \dot{\bar{A}}^{\mathrm{T}}_{ij} = -\frac{1}{4\pi c^2}\epsilon^{\alpha}_{ij}\lim_{\omega\to 0}[\omega a^{\dagger}_{\alpha}(\omega, \boldsymbol{n}) + \mathrm{c.c.}].$$

This equation relates the memory to the creation and annihilation of a fractonic soft (zero frequency) photon. As a result, an amplitude with an external soft photon factorizes according to (33) with soft factors

$$S^{\blacksquare}_{ij} = -4\pi c^2_{\#}\delta\bar{A}^{\blacksquare}_{ij}. \tag{34}$$

Thus, the soft factors $S^{\mathrm{T}}_{ij}, S^{\mathrm{TL}}_{ij}$ in (33) are given up to overall factors $-4\pi c^2_{\blacksquare}$ by (14a) and (14b), respectively. An alternative derivation of the soft factors is detailed in Supplemental Material [53] and involves the use of Feynman diagrams within a simple effective model that describes the dynamics of dipoles coupled to the fracton gauge field.

Discussion and outlook.—In this work, we introduced an observable double kick memory effect (see section on decoupled field equations and memory effect) and the corresponding dipole soft theorem (see section on soft factors from the memory effect), which we related to the asymptotic symmetries of fracton gauge theory. This provides the first instance of an IR triangle for a theory beyond Lorentz symmetry and further evidence for the robustness of this triangular correspondence.

The tools developed and implemented in this work can be used to study radiation and IR effects beyond Lorentzian symmetries, which opens the door to explore other models of relevance to condensed matter systems and beyond, e.g., [45,46,56–83]. The double kick effect (see Fig. 1) is an example of a novel memory observable which exhibits the more intricate structure that can appear in a non-Lorentzian theory.

One avenue we would like to highlight involves leveraging dualities in $2+1$ dimension between (specific models of) fractons and vortices [84] as well as the relationship with elasticity [85,86] (see [87] for a review). These dualities open up exciting possibilities for creating experimental setups that are potentially easier to realize and which could facilitate the observation of memory effects (see also [88]).

While fractons have originated from condensed matter physics, they might play an important role in the holographic understanding of gravity in asymptotically flat spacetimes. The reason is the correspondence between the fracton algebra and the Carroll algebra which is the underlying symmetry of gravity in asymptotically flat spacetimes [9–11,89–91]. Indeed, many of the discussed dualities and experiments can equally well be seen though the lens of Carrollian physics, e.g., in [92] insights from fractons have been used in the context of Carroll fluids.

We have focused on the leading IR behavior. However, it might be interesting to explore subleading effects and consider how celestial holography [17–21] could be extended to this setup. After all Lorentz symmetry is absent, but we still recover an analog IR structure. In addition, to getting closer to experiments that investigate dipole symmetry [41–44] it might also be interesting to introduce and study boundaries at finite distances.

Motivated by the historically prolific interdisciplinary dialogue between high-energy and condensed matter physics, we are also excited by the prospect that the inaugural experimental validation of memory effects may manifest within the domain of condensed matter systems. We hope that this work will serve as an initial stepping stone for this promising endeavor.

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